Development of Convoluted Proportional Hazard Model with Constrained Parameters with Application to Recidivism Data

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Abstract
Classical survival models usually struggle to simultaneously capture the dual dynamics of hazard namely: short term and long term risk within a unified framework. Although the Cox proportion hazard model provide flexibility through its semi-parametric structure but it fails to distinctively distinguish between transient and persistent risk mechanism. In contrast, fully parametric models such as Weibull distribution give structural interpretation but lack the flexibility of covariate-driven hazard evolution. This study develops a convoluted proportion hazard model that integrates this complementary strength into a single framework. The developed model combines a Cox proportional hazard component with a parametric Weibull baseline structure and with constrained parameters p and q where $p,\ q\ \varepsilon\ [0,1]$ thereby enabling explicit decomposition of short and long term hazard contributions. Estimation were done using maximum likelihood estimation, procedures parameter are derived explicitly including closed-form expressions for the log-likelihood function and its corresponding score equations. Simulation was conducted to estimate the parameters and model performance is evaluated using standard information criteria and likelihood based comparison measures. Finally, the empirical analysis demonstrates that the developed model outperform the traditional models and fuilt parametric alternatives. The model offers a clear interpretation of risk dynamics. The new model contributes to survival methodology by introducing a structure yet flexible approach capable of disentangling short term and long term hazard effect within a unified proportional hazard setting.

Keywords:
Convoluted Hazard Model, Simulation, Maximum Likelihood Estimation Regression, Cox Regression

Introduction

Survival analysis has found broad applicability across diverse disciplines, for instance, it is employed in engineering to estimate the time until machine failure, in social sciences it explore the duration of marriages, and in finance it evaluate the time before stock prices decline. Weibull distribution, a generalization of the exponential distribution, offers great flexibility by allowing for non-constant hazard rates. In contrast, the Cox model is widely favored for its robustness, minimal distributional assumptions, and capacity to handle non-negative hazard ratio across covariate groups. This study focuses on these two models, combining their strengths into a novel hybrid framework to analyze recidivism rate more flexibly and accurately. The Cox model, by virtue of its semi-parametric formulation, provided researchers with a flexible tool to model time-to-event data without making strict assumptions about the baseline hazard function. However, its flexibility has proven insufficient in many

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real-world applications, especially where complex hazard dynamics such as dual risk, long-term survivorship, or structural heterogeneity are present. This has motivated numerous methodological extensions in recent decades, for example, Hannal *et al.* (2021) proposed shape-constrained Cox models via the CPH-shape algorithm, enabling the baseline hazard to be estimated under monotonic, unimodal, or U-shaped restrictions. Faweya *et al* (2026) convolute cox-weibull and proportional hazard model with constrained parameters for modeling short-term and long-term survival risks Eni *et al.* (2022) extended survival modeling through mixture cure models. Begun *et al*. (2023) advanced the field by introducing double-Cox frailty models, embedding two Cox processes with shared frailty. Kızılaslan *et a*l. (2024) investigated a Weibull mixture cure frailty model with elastic-net penalization to handle high-dimensional covariates. Lai (2014) investigated the generalized Weibull distribution, using the traditional Weibull as a foundational case. {Sarhan and Zaindin (2009)} proposed the modified Weibull distribution, which offered enhanced flexibility to model increasing, decreasing, and constant hazard rates beyond what the standard Weibull could achieve. Gusmao *et al*. (2009) examined the generalized inverse Weibull (GIW) distribution, Nikulin and Haghighi (2006) further extended the Weibull by introducing an additional shape parameter, creating the generalized power Weibull (GPW) distribution. Oguntunde *et al*. (2015) formulated a four-parameter model known as the exponentiated generalized Weibull (EGW) distribution, Selim and Badr's (2016) Kumaraswamy generalized power Weibull distribution, Pu *et al*.'s (2016) generalized class of exponentiated modified Weibull models, Selim (2018) with the generalized power generalized Weibull distribution, and Broderick *et al*. (2020), who developed the exponentiated generalized power series family of distributions. Aboukhamseen *et al*. (2016) introduced the proportional hazard inverse Weibull distribution, the log-rank test statistic is a chi-square measure based on the difference between observed and expected events.

Methodology

Survival Analysis is a collection of statistical procedures for data analysis for which the outcome variable of interest is time until an event occurs. The survival function denoted by S(t) gives the probability that a person survives longer than some specified time t. S(t) gives the probability thsat random variable T exceed the specified time t.

The hazard function denoted by h(t), is given by the formula: h(t) equals the limit, as

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t \le T < t+\Delta t / T \ge t)}{\Delta t} \qquad (1)$$

The hazard function $h(t)$ gives the instantaneous potential per unit time for the event to occur, given that the individual has survived up to time t.

More generally, the relationship between $S(t)$ and $h(t)$ can be expressed equivalently in either of two calculus formulae as

$$S(t) = \exp[-\int_0^1 h(u)\,du]$$

And,

$$h(t) = -\frac{dS(t)/dt}{S(t)}$$

3.1.5 **Hazard Ratio (HR)**

Hazard ration (HR) is defined as the hazard for one individual divided by the hazard for a different individual. The two individual being compare can be distinguished by their values for the set of predictors, that is the x_1's

$$\hat{H}R = \frac{\hat{h}(t, x^*)}{\hat{h}(t, x)}$$

Where $\hat{H}R$ is the Hazard Ratio Estimate

$x^* = x_1^*, x_2^* \ldots x_p^*$ and $x = x_1, x_2, \ldots, x_p$

The groups are typically coded so that group with the larger hazard corresponds to x^*, and the group with the smaller hazard corresponds to x. The group with larger hazard is coded 1 and group with smaller hazard is coded 0.

$$\hat{H}R = \frac{\hat{h}(t, x^*)}{\hat{h}(t, x)} = \frac{\cancel{h_0(t)}e^{\sum_{i=1}^{p}\beta_i^* x_i^*}}{\cancel{h_0(t)}e^{\sum_{i=1}^{p}\beta_i^{\wedge} x_i}} = e^{\sum_{i=1}^{p}\beta_i^{\wedge}(x_i^* - x_i)}$$

$$\hat{H}R = e^{\sum_{i=1}^{p}\beta_i^{\wedge}(x_i^* - x_i)}$$

(6)

3.1.6 Cox Regression Model

. The model is given as;

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$$h(t, x) = h_o(t)\exp(\sum_{i=1}^{p}\beta_i x_i) \quad (7)$$

Where $h(t, x_i)$ = the hazard function at time t for a subject with covariate values

$x_1, x_2, ..., x_p$

$h_o(t)$ = the baseline hazard function i.e the hazard function when all covariates equal zero

x_i = the i^{th} covariate in the model and

β_i = the regression coefficient for the i^{th} covariate x_i.

Estimation of the Coefficient of the Cox's Regression Model

To estimate the coefficient $\beta_1, \ldots, \beta p$ Cox suggest a maximum likelihood procedure where the likelihood function is based on a conditional probability of failure. We have that t(1)<t(2)<…<t(k) are the k exact failure times and R(t(i)) is the risk set of patient(convict) at time t(i). For the particular failure at time t(i), conditional on the risk set R{t(i)}, the probability that the failure is on the individual is

$$\frac{\exp(\sum_{j=1}^{p}\beta_j x_{ji})}{\sum_{1\in R\left(t_{(1)}\right)} exp(\sum_{j=1}^{p}\beta_j x_{ji})} \quad (8)$$

Each failure contributes a factor and hence the conditional log-likelihood is

$$LL(\beta) = \sum_{i=1}^{k}\sum_{j=1}^{p}\beta_j j_i - \sum_{i=1}^{k} log_e\left[\sum_{i\in R(t_{(i)})}\exp(\sum_{j=1}^{p}\beta_j x_{ji})\right]$$
(9)

Maximum likelihood estimate of β's are obtained by solving simultaneously the P equation that are derivative of LL(β) with respect to β1,…βp respectively, equating to zero. The P equations are

$$U(\beta_1 \ldots.\beta_P) = \sum_{i-1}^{k}\left[X_{ui} - A_{ui}(\beta_1 \ldots.\beta_p)\right] = 0, \ U = 1, \ldots, p \quad (10)$$

Where

$$A_{ui}\left(\beta_1, \ldots., \beta_p\right) = \frac{\sum_{1\in R(t_{(1)})} x_{ul}\, exp(\sum_{j=1}^{p}\beta_j x_{ji})}{\sum_{1\in R(t_{(1)})} exp(\sum_{j=1}^{p}\beta_j x_{ji})} \quad (11)$$

The P equations can be solved numerically by the Newton-Raphson method of iteration where the estimate of the coefficients are obtained by iteration use of U(β1,β2,…βk) and the second derivative of LL(β):

$$luv(\beta_1,\ldots,\beta_p) = -\sum_{i=1}^{k} C_{(uvi)}\left(\beta_i, \ldots.\beta_p\right) \quad (12)$$

Where

$$C_{[uvi]}\left(\beta_i \ldots.\beta_p\right) = \frac{\sum_{i\in R(t)} x_{ul}\, x_{vl}\, exp\left[\sum_{j=1}^{p}\beta_j x_{j1}\right]}{\sum_{i\in R(t)} exp\sum_{j=1}^{p} b_j x_{j1}} = -A_{ui}\left(\beta_{i,} \ldots.\beta_p\right)A_{vi}\left(\beta_{i,} \ldots.\beta_p\right) \quad (13)$$

In addition to the values of the estimated regression coefficient β1, β2,…βk give the proportional change that can be expected from the hazard, related to changes in the covariate. If these values are positive, the prognosis is worse for subject with higher values of that covariate, while if the values are negative the prognosis is better for subject with higher values of that covariate. In the particular case where the survival times involve ties (which is a more common problem in practice than the case of continuous survival time) Cox proposed a more general model. This model generalizes

$$h(t/x) = h_o(t)\exp(\beta_1 x_1 + \beta_2 x_2 + \cdots. + \beta_p x_p) = h_o(t)\, exp\sum_{j=1}^{p} b_j\, x_j \quad (14)$$

To discrete time by a logistic transformation

$$\frac{h_{(t)}dt}{1-h_{(t)}dt} = \frac{h_o(t)dt}{1-h_o(t)dt}\, exp\sum_{j=1}^{p} b_j\, x_j$$

(15)

Suppose that among the survival times t1,t2…,tn there are k distinct failure times t(1) < . . . < t(k)

Let M(i)= the multiplicity of t(i), M(i) > 1 if there is more than one observation with value t(i),

M(i)=1, if there is only one observation with value t(i)

Let r{t(i)} denote the set of individual at risk at time t(i), and r(i) be the number of such individual. At time t(i), the probability that the individual fails as observed conditionally on the risk set R{t(i)} is

$$\frac{\exp(\beta_1 z_{1i}+\beta_2 z_{2i}+\cdots+\beta_p z_{pi})}{\sum_{1\in R\left(t_{(1)}\right)} exp\,(\beta_1 z_{1i}+\beta_2 z_{2i}+\cdots+\beta_p z_{pi})} \quad (16)$$

Where Z_{1i} is the sum of x1i's over the M(i) individual failing at t(1)…

Z_{pi} is the sum of x_{pi}'s over the M(i) individual failing at t(i)

The condition likelihood function, function U and I are,

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$LL(\beta) = \sum_{i=1}^{k}(\beta_1 z_{1i} + \beta_2 z_{2i} + \cdots + \beta_p z_{pi}) - \sum_{i=1}^{k} log\left[\sum_{i\in R(t_{(i)})} \exp(\beta_1 z_{1i} + \beta_2 z_{2i} + \cdots + \beta_p z_{pi})\right]$ (17)

$U(\beta_i \ldots . \beta_P) = \sum_{i-1}^{k}[Z_{ui} - m_{(1)}A_{ui}(\beta_1 \ldots . \beta_p)] = 0,\ U = 1, \ldots, p$ (18)

luv($\beta_1,\ldots,\beta_p$)= $\frac{m_{(i)}[r_{(i)}-m_{(i)}]}{r_{(i)}-1} C_{(uvi)}(\beta_i, \ldots . \beta_p)$ (19)

3.1.9 Weibull Regression Model

The Weibull distribution is characterized by its probability density function (PDF):

$f(t:\alpha,\beta) = \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1} exp\left(-\left(\frac{t}{\gamma}\right)^{\alpha}\right)$: Where $\alpha,\gamma>0$

$t \geq 0$ (20)

Where α is the shape parameter, γ is the scale parameter and t represent time- to- event variable

3.2.0 Estimation of Weibull Parameters

$f(t:\alpha,\beta) = \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1} exp\left(-\left(\frac{t}{\gamma}\right)^{\alpha}\right)$: Where $\alpha, \gamma > 0$

$t \geq 0$

Where α is the shape parameter and β is the scale parameter

The cumulative distribution function (CDF); = $F(t) = 1 - \exp(-\left(\frac{t}{\gamma}\right)^{\alpha})$ (21)

Survival Function = $S(t) = F(t) = 1 - \exp(-\left(\frac{t}{\gamma}\right)^{\alpha})$ (22)

Hazard Function = $h(t) = \frac{f(t)}{S(t)} = \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}$ (23)

Increasing hazard if α > 1

Constant hazard if α = 1

Decreasing hazard if α < 1

Parameter estimation via maximum likelihood estimation (MLE)

Given n i.i.d, lifetimes $t_1, \ldots, t_n$

shape parameter α > 0

scale parameter $\gamma > 0$

P.D.F is $f(t:\alpha,\beta) = \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1} exp\left(-\left(\frac{t}{\gamma}\right)^{\alpha}\right)$ (24)

The likelihood is $L(\alpha,\gamma) = \prod_{i=1}^{n} f(t:\alpha,\gamma)$ (25)

Take the log of $L(\alpha,\gamma)$ = logL = $\sum_{i=1}^{n}[\log\alpha - log\gamma + (\alpha-1)log\text{t} - \left(\frac{t}{\gamma}\right)^{\alpha}$ (26)

logL = $nlog\alpha - nlog\gamma + (\alpha-1)\sum_{i=1}^{n} logt - \sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha}$ (27)

Define $\sum_{i-1}^{n} t$ = L and $\sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha}$ = Z (28)

So, $logL = nlog\alpha - nlog\gamma + (\alpha-1)L - Z$ (29)

$We\ differentiate\ logL\ with\ respect\ to\ \alpha\ and\ \gamma, now\ with\ respect\ to\ \gamma, we\ have;$

Z = $\sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha} = \sum_{i=1}^{n} t^{\alpha}\gamma^{-\alpha}$ then (30)

$\frac{\partial z}{\partial \gamma} = -\alpha\gamma^{-\alpha-1}\sum_{i=1}^{n} t^{\alpha} = \frac{-\alpha}{\gamma} Z$ (31)

$\frac{\partial l}{\partial \gamma} = \frac{-n}{\gamma} + \frac{\partial z}{\partial \gamma} = \frac{-n}{\gamma} + \frac{\alpha}{\gamma} Z = \frac{\alpha z - n}{\gamma}$ (32)

With respect to α

$L(P) = nlog\alpha - nlog\gamma + (p-1)L - \sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha}$ (33)

$\frac{\partial}{\partial\alpha}$(nlog$\alpha$) = $\frac{n}{\alpha}$, $\frac{\partial}{\partial\alpha}(-nlog\gamma)$ = 0, $\frac{\partial}{\partial\alpha}((p-1)L)$ = L (34)

$\frac{\partial}{\partial\alpha}(\sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha}) = \sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha}\log(\frac{t}{\gamma})$ (35)

$\frac{\partial l}{\partial \alpha} = \frac{n}{p} + L - \sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha}\log(\frac{t}{\gamma})$ (36)

MLE

We set the partial derivative to zero

From (1) $\frac{\partial l}{\partial \gamma} = 0$, $\frac{\alpha z - n}{\gamma}$ =0, Z = $\sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha} = \frac{n}{\alpha}$ (37)

(2) $\frac{\partial}{\partial\alpha} = 0$, $\frac{n}{p} + \sum_{i-1}^{n} t - \sum_{i=1}^{n}\left(\frac{t}{\gamma}\right)^{\alpha}\log(\frac{t}{\gamma}) = 0$ (38)

3.2.2 Convoluted hazard model

In this study, the goal is to create a combine hazard function that integrates the Weibull distribution and the Cox proportional hazard model. This combine model will account for both baseline hazard shapes (through the

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Weibull distribution) and covariates effects (through the Cox model).
The Weibull distribution is characterized by its probability density function (PDF):
$f(x:\alpha,\beta) = \frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{\alpha-1} exp\left(-\left(\frac{x}{\gamma}\right)^{\alpha}\right)$:
Where α , γ > 0 (39)

x ≥ 0
Where α is the shape parameter of the Weibull distribution and β is the scale parameter of the Weibull distribution. From this, we derive the Weibull hazard function, which describes the instantaneous failure rate at time t:

$hw(t,\alpha,\gamma) = \frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{\alpha-1}$
(40)

The Cox proportional hazards model, a semi-parametric model has no probability density function (PDF) but expresses the hazard function as a product of a baseline hazard function and a term that account for covariates

$h(t,x_i) = h_0(t)\exp(\sum_{i=1}^{k}\beta_i x_i)$
(41)
$h_0(t)$ is the baseline hazard function,
x_i are the covariates,
B_i are the coefficients associated with each covariate.

3.2.3 Combined Hazard Function

To combine these models, we introduce weight p and q such that p + q =1, where
0 ≤ p, q ≤ .These weight will control the contribution of each model to the combine hazard function. The combine hazard function is then given by:

$g(t,x_i) = p.h(t,x_i) + q.hw(t,\alpha,\gamma)$

(42Substituting (2) and (3) into (4)
$g(t,x_i) = p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i + q.\frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{\alpha-1}$
(43)
Using p = 1 − q , hence
$g(t,x_i) = p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i + (1-\text{p}).\frac{\alpha}{\gamma}$
(44)

3.2.3 Maximum Likelihood Estimation for the Combined Weibull-Cox Model

From the combined hazard function (6), we have
$h(t,x_i) = p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i + (1-\text{p}).\frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{\alpha-1}$ (45)
Define cumulative hazard function of the form
$H(t,x_i) = \int_0^t h(u,x_i)\,d$ (46)

And a survival function
$S(t,x_i) = \exp(-H(t,x_i))$ (47)

Given n observations with survival times $t_{i,}$ covariates x_i and censoring indicatorδ_i, the likelihood function id given by

$L(\alpha,\gamma,\beta_i,p,t_{i,}x_i,\delta_i) = \prod_i^n h(t_{i,}x_i)^{\delta_i}.S(t_{i,}x_i)$
(48)
Taking the natural logarithm of the likelihood function:
Log L = $\sum_{i=1}^{n}\{\ \delta_i \log h(t_{i,}x_i) - H(t_{i,}x_i)\}$
(49)

$Log\ L(\alpha,\gamma,\beta_i,p,) = \sum_{i=1}^{n}\left[(\delta_i \log p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i) + (\left(\frac{x}{\gamma}\right)^{\alpha-1})\frac{\alpha}{\gamma}\ (1-\text{p}). - \sum_{i=1}^{k}\int_0^t h(u,x_i)du\ .\right]$
(50)
We now differentiate this with respect to α, compute $\frac{\partial logL}{\partial\alpha}$
Take the derivative term by term, term1: δ_i log $h(x_i\ ,t_i)$, only the Weibull component depends on α, then; $h(x_i,t_i)$ = P. $h_{cox}(x_i,t_i)$ + (1-p). $\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}$

(51)
let $h_i = h(x_i,t_i)$, $h_{wi} = \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}$
(52)
$\partial h_i/\partial\alpha = (1-p).\ \partial\, h_{wi}/\partial\alpha$ (53)
we differentiate h_{wi} i.e. $\partial/\partial\alpha[\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}]$ by using product rule, we have
$\partial h_i/\partial\alpha = (1-p).[\frac{1}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1} + \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}\log(\frac{t}{\gamma})]$
(54)
Applying chain rule; $\frac{\partial}{\partial\alpha}\ log\ h(x_i,t_i) = 1/h_i\ .\ \partial h_i/\partial\alpha$
so the first term becomes $\delta i \sum_{i=1}^{n}(\frac{1}{h(t;x)}\ .\ \frac{\partial h(t;x)}{\partial\alpha})$ (55)

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substituting in $\sum_{i=1}^{n} \delta i \left(\frac{(1-p)}{h(t;x)}\right. . \left[\frac{1}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1} + \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}\log\left(\frac{t}{\gamma}\right)\right]\Big)$ (56)

factor out common parameters: $\sum_{i=1}^{n} \delta i \left(\frac{(1-p)}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}. \left[\frac{1}{h(t;x)} + \frac{\alpha \log(\frac{t}{\gamma})}{h(t;x)}\right]\right)$ (57)

$$\frac{\partial Log\ L}{\partial \alpha} = \sum_{i=1}^{n}\left[\left(\delta_i.\frac{(1-p).\frac{\alpha}{\gamma}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}.\log\left({}^{t_i}/_{\gamma}\right)}{p.h_0(t)exp\sum_{i=1}^{k} B_i x_i+(1-p).\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}}\right]\right. - \sum_{i=1}^{n}\int_0^{t_i}\frac{\partial h(u,x_i)}{\partial \alpha}du \quad (58)$$

Differentiate with respect to γ

$$\frac{\partial \log}{\partial \beta} = \sum_{i=1}^{n}\left[\delta i \frac{1}{h(t;x)} . \frac{\partial h(t;x)}{\partial \gamma} - \int_0^t \frac{\partial h(u;x)}{\partial \gamma} du \right. \quad (59)$$

Recall that the mixture is $h(t;x)$ = P. $h_o(t)$ exp $(\sum_{i=1}^{n}\beta_i x_i)$ + (1-p). $\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}$

Only the Weibull part depends on γ

$H_w(t) = \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1} = \gamma^{-\alpha}.t^{(\alpha-1)}$,

we differentiate this w.r.t γ using product and chain rule

$\frac{\partial}{\partial\gamma}\left[\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}\right]$. Let $f(\gamma) = \frac{\alpha}{\gamma}$ and $g(\gamma) = \left(\frac{t}{\gamma}\right)^{\alpha-1}$ (60)

Then, $\frac{\partial}{\partial\beta}(fg) = f'g + fg'$ (61)

$f' = \frac{\alpha}{\gamma 2}$, $g = \left(\frac{t}{\gamma}\right)^{\alpha-1}$ and $g' = (\alpha-1)\left(\frac{t}{\gamma}\right)^{\alpha-2}$. $\frac{1}{\gamma 2} = -(\alpha-1).\left(\frac{t}{\gamma}\right)^{\alpha-1}.\frac{1}{\gamma}$ (62)

$\frac{\partial}{\partial\gamma}h_w(t) = -\frac{\alpha}{\gamma 2}.\left(\frac{t}{\gamma}\right)^{\alpha-1} - \frac{\alpha}{\gamma}.(\alpha-1)\left(\frac{t}{\gamma}\right)^{\alpha-1}.\frac{1}{\gamma}$ (63)

Factor common terms

$\frac{\partial}{\partial\gamma}h_w(t) = -\left(\frac{t}{\gamma}\right)^{\alpha-1}. \frac{1}{\gamma}\left[\frac{\alpha}{\gamma}+\frac{\alpha(\alpha-1)}{\gamma}\right] =-$ (1-p) $\left(\frac{t}{\gamma}\right)^{\alpha-1}.\frac{1}{\gamma}\left[\frac{\alpha}{\gamma}+\frac{\alpha(\alpha-1)}{\gamma}\right]$ (64)

$$\frac{\partial Log\ L}{\partial \gamma} = \sum_{i=1}^{n}\left[\left(\delta_i.\frac{(1-p).\frac{1}{\gamma}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}.\left((\alpha-1)\frac{1}{\gamma}\right)-\frac{\alpha}{\gamma^2}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}}{p.h_0(t)exp\sum_{i=1}^{k} B_i x_i+(1-p).\frac{\alpha}{\gamma}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}}\right]\right. - \sum_{i=1}^{n}\int_0^{t_i}\frac{\partial h(u,x_i)}{\partial \gamma}du \quad (65)$$

To get β_i, we differentiate $h(t;x)$ with respect to β_i

$$\frac{\partial \log}{\partial \beta i} = \sum_{i=1}^{n}\left[\delta i \frac{1}{h(t;x)} . \frac{\partial h(t;x)}{\partial \beta i} - \int_0^t \frac{\partial h(u;x)}{\partial \beta i} du\right.$$

$h(t;x)$ = = P. $h_o(t)$ exp $(\sum_{i=1}^{n}\beta_i x_i)$ + (1-p). $\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}$

only the Cox part depends on β_i, so we have;

$\frac{\partial h(t;x)}{\partial \beta i}$ = P. $h_o(t)$ exp $(\sum_{i=1}^{n}\beta_i x_i)$. x_i, we plug in to first term (66)

$\delta_i . \frac{1}{h(t;x)}$. P. $h_o(t)$ exp $(\sum_{i=1}^{n}\beta_i x_i)$. x_i (67)

=

$$\frac{\partial Log\ L}{\partial \beta_i} = \sum_{i=1}^{n}\left[\left(\delta_i.\frac{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i\ .\ .x_i}{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i+(1-p).\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}}\right]\right. - \sum_{i=1}^{n}\int_0^{t_i}\frac{\partial h(u,x_i)}{\partial \beta_i}du \quad (68)$$

To get P, we differentiate $h(t;x)$ with respect to P

$$\frac{\partial \log L}{\partial p} = \sum_{i=1}^{n}\left[\delta i \frac{1}{h(t;x)} . \frac{\partial h(t;x)}{\partial p} - \int_0^t \frac{\partial h(u;x)}{\partial p} du\right.$$

But $h(t;x) = P.h_o(t)\ exp\ (\sum_{i=1}^{n}\beta_i x_i) + (1-p).\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}$

$\frac{\partial h(t;x)}{\partial p}$ = $h_o(t)$ exp $(\sum_{i=1}^{n}\beta_i x_i)$ - $\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}$ (69)

$$\frac{\partial Log\ L}{\partial p} = \sum_{i=1}^{n}\left[\left(\delta_i.\frac{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i - \frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}}{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i+(1-p).\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}}\right]\right. - \sum_{i=1}^{n}\int_0^{t}\frac{\partial h(u,x_i)}{\partial p}du \quad (70)$$

Setting (12), (13), (14) and (15) to zero, that is Setting $\frac{\partial Log\ L}{\partial \alpha} = \frac{\partial Log\ L}{\partial \gamma} = \frac{\partial Log\ L}{\partial \beta_i} = \frac{\partial Log\ L}{\partial p} = 0$ (71)

$$\frac{\partial Log\ L}{\partial \alpha} = \sum_{i=1}^{n}\left[\left(\delta_i.\frac{(1-p).\frac{1}{\gamma}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}.\log\left({}^{t_i}/_{\gamma}\right)}{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i+(1-p).\frac{\alpha}{\gamma}\left(\frac{x}{\gamma}\right)^{\alpha-1}}\right]\right. - \sum_{i=1}^{n}\int_0^{t_i}\frac{\partial h(u,x_i)}{\partial \alpha}du = 0 \quad (72)$$

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$$\frac{\partial Log\ L}{\partial \gamma} = \sum_{i=1}^{n}\left[\left(\delta_i . \frac{(1-p).\frac{\alpha}{\gamma}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}.\left((\alpha-1)\frac{1}{\gamma}\right) - \frac{\alpha}{\gamma^2}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}}{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i + (1-p).\frac{\alpha}{\gamma}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}}\right]\right. - \sum_{i=1}^{n}\int_0^{t_i}\frac{\partial h(u,x_i)}{\partial \gamma}du = 0 \qquad (73)$$

$$\frac{\partial Log\ L}{\partial \beta_i} = \sum_{i=1}^{n}\left[\left(\delta_i . \frac{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i\ .\ x_i}{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i + (1-p).\frac{\alpha}{\gamma}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}}\right]\right. - \sum_{i=1}^{n}\int_0^{t_i}\frac{\partial h(u,x_i)}{\partial \beta_i}du = 0 \qquad (74)$$

$$\frac{\partial Log\ L}{\partial p} = \sum_{i=1}^{n}\left[\left(\delta_i . \frac{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i - \frac{\alpha}{\gamma^2}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}}{p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i + (1-p).\frac{\alpha}{\gamma}\left(\frac{t_i}{\gamma}\right)^{\alpha-1}}\right]\right. - \sum_{i=1}^{n}\int_0^{t_i}\frac{\partial h(u,x_i)}{\partial p}du = 0 \qquad (75)$$

The convoluted proportional hazard model with constrained parameters has been structured in a way that facilitates its computational implementation using standard software such as R, Python, and Stata.

Simulation Study and Analysis

The results of the simulation study was conducted and the real-life recidivism dataset on inmates in Ado-Ekiti custodial Center, Ekiti State from 2014-2025. The analysis compares the performance of four competing models—Cox Proportional Hazards (CoxPH), Double-Cox, Weibull, and the Proposed Convoluted Proportional Hazard model across different sample sizes. Calculation on prevalence rate of recidivism in the custodial center, the result is 31.9 which means almost one-third of the population are recidivist. At sample size n = 25, it was observed that the new convoluted proportional hazard model outperformed other existing model having lowest value in AIC and BIC showing robustness even with limited data. At n = 50, the double cox model performed better than other models, at n = 75, the new model again outperform other existing models by the result from AIC and BIC. At n = 100, and 200 the convoluted proportional hazard model shows superior power over existing models. In the real life data of sample size 257, the new hybrid model also outperform other existing model in both AIC and BIC even log-likelihood shows supremacy of the new model. It was observed that marital status, offence category and employment status are the contributory factors to recidivism, as they are all significant with values 0.0095, 0.0001 and 0.0000 respectively. The shape parameter is greater than one (1.5262) showing hazard increases over time that is recidivism risk grows with time. The scale parameter indicates short hazard time. To distinguish distinctively the dual dynamic of hazard is one of the focus of this study. The results are interpreted based on parameter estimates, standard errors, significance levels, and information criteria (AIC and BIC). Model performance is also examined using the Kaplan–Meier survival and kernel-smoothed hazard functions to demonstrate the underlying survival and hazard structures in the data

Table 4.1 Prevalence Rate of Recidivism

Total number of inmates (a)	**805**
No of re-incarcerated inmates (b)	**255**
% Prevalence Rate = a/b*100	**31.68**

Interpretation: The Table 4.1 above shows that the prevalence rate of recidivism is 31.68 which means almost one-third of the population are recidivist.

Simulation Studies**:-**Below are the simulation studies on our new convoluted proportional hazard model and other existing models:

$$Log\ L(\alpha,\gamma,\beta_i,p,) = \sum_{i=1}^{n}\left[\left(\delta_i \log p.h_0(t)exp\sum_{i=1}^{k}\beta_i x_i\right) + \right.$$

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$$\left(\left(\frac{x}{\gamma}\right)^{\alpha-1}\right)\frac{\alpha}{\gamma}\ (1-\ \mathrm{p}).-\sum_{i=1}^{k}\int_{0}^{t}h(u,x_i)du\right]$$

$\widetilde{H}(t|u,Z) = ZH(t|u) =$ $Ze^{\beta_{scale}u}\left(\frac{t}{a}\right)^{bexp(\beta_{shape}u)}$ as the case of Weibull and as

$\widetilde{H}(t|u,Z) = ZH(t|u) =$ $Ze^{\beta_{scale}u}\frac{\left(e^{bexp(\beta_{shape}u)t}-1\right)}{bexp(\beta_{shape}u)}$ in the case of Gompertz model. Where:

a, is the scale parameter of the baseline survival distribution

b, is the shape parameter of the baseline survival distribution

u, is the vector column of the covariates

β_{shape} (β_{sh}) and β_{scale} (β_s), are the vector rows of the Cox regression parameter

Z, is the gamma-distributed frailty term

Estimation of Coefficient with initial values include: $shape\ (\alpha) = 1.5$, $Scale\ (\gamma) = 2.0$, $\beta_1 = 0.1$, $\beta_2 = -0.6$, $\beta_3 = 0.2$, $\beta_4 = -0.2$, $delta\ (\delta) = 1.0$, and $p = q = 0.5$

Table 4.2 Simulation study on n = 25

N	Model/Parameter	Coefficient	Standard Error	Prob	AIC	BIC
25	Coxph	$\beta_1 = 1.5313$ $\beta_2 = 0.3524$ $\beta_3 = 0.8377$ $\beta_4 = 0.9494$	0.6530 2.8380 1.1940 1.0530	2.4610 0.7246 1.2500 1.6816	110.2777	115.1532
	Weibull	$\beta_0 = 0.0158$ $\beta_1 = -0.2669$ $\beta_2 = 0.7570$ $\beta_3 = 0.1212$ $\beta_4 = 0.0385$ Log(Scale)$= -0.3210$ Scale $= 0.7250$	0.2095 0.1584 0.2283 0.1375 0.2019 0.1620	0.9398 0.0919 0.0009 0.3779 0.8490 0.0810	33.3072	40.620
	Double-Cox Weibull	$\mathrm{Log}_a = 0..6064$ $\mathrm{Log}_b = 0..4060$ $\beta s_0 = 0.0910$ $\beta s_1 = 0.0910$ $\beta s_2 = -0.1016$ $\beta s_3 = 0.0910$ $\beta s_4 = -0.1001$ $\beta sh_0 = 0.1005$ $\beta sh_1 = -0.1004$ $\beta sh_2 = 0.0462$ $\beta sh_3 = 0.0097$ $\beta sh_4 = -0.0185$	NaN NaN 14.0580 2.9785 0.2727 2.6826 5.5191 0.0251 0.1022 0.0030 NaN NaN	NaN NaN 0.9940 0.9730 0.7100 0.9700).9860 0.0124** 1. 25e-06*** 0.0231 NaN NaN	86.5626	101.1891
	Double Cox Gompertz	$\mathrm{Log}_b = 5..2007$ $\beta s_0 = -8.0567$ $\beta s_1 = -3.4227$ $\beta s_2 = -0.2790$ $\beta s_3 = 1.2514$ $\beta s_4 = 15.1973$ $\beta sh_0 = -5.0795$ $\beta sh_1 = 0.2249$ $\beta sh_2 = 0.0444$ $\beta sh_3 = 0.2893$ $\beta sh_4 = -1.6224$	0.0000 NaN 1.0073 0.0604 1.5717 NaN NaN 0.0735 0.0054 0.1830 NaN	0.0000 NaN 0.0007*** 1.32e-05*** 0.4259 NaN NaN 0.0022** < 2e-16*** 0.1139 NaN	32.1755	45.5831
	Proposed Convoluted Proportional Hazard	$\alpha = 1.8374$ $\gamma = 0.9567$ $\beta_1 = 0.1000$ $\beta_2 = -0.6997$ $\beta_3 = 0.1999$ $\beta_4 = -0.2001$ $\delta = 0.0083$ $p = 0.0024$	0.2930 0.0000 0.1963 0.2892 0.2567 0.1430 10.6327	3.5e-10*** 0.0000 0.6105 0.0155* 0.4361 0.1617 0.7453	32.0691	41.8201

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Interpretation: At the smallest sample size (n = 25), the proposed Convoluted Proportional hazard model achieved an AIC of 32.07 and a BIC of 41.82, outperforming the CoxPH (AIC = 110.28), Double-Cox (AIC = 86.56, 32.17), and Weibull (AIC = 33.31) models. The hybrid's superior performance, even with limited data, suggests robustness in capturing both short- and long-term risks. The mixture parameters (α = 1.8374, γ = 0.9567, p = 0.0024, δ = 0.0083) reflect its flexible structure that accommodates both proportional hazards and parametric components simultaneously.

Table 4.3 Simulation study on sample size 50

N	Model/Parameter	Coefficient	Standard Error	Prob	AIC	BIC
50	Coxph	$\beta_1 = 1.6759$ $\beta_2 = 0.5123$ $\beta_3 = 1.0592$ $\beta_4 = 0.6626$	0.1611 0.1455 0.1540 0.1891	0.0014** 4.3e-06*** 0.0173* 0.2995	281.2769	288.9249
	Weibull	$\beta_0 = 0.0275$ $\beta_1 = -0.4210$ $\beta_2 = 0.5377$ $\beta_3 = -0.3231$ $\beta_4 = 0.3173$ Log(Scale)= -0.2883 $Scale = 0.7500$	0.1149 0.1146 0.1283 0.1092 0.1350 0.1143	0.8108 0.0002 2.8e-05 0.0031 0.0188 0.0117	116.7042	128.1764
	Double-Cox Weibull	$Log_a = 0..6117$ $Log_b = 0..4063$ $\beta s_0 = 0.0998$ $\beta s_1 = 0.0999$ $\beta s_2 = -0.1029$ $\beta s_3 = 0.0999$ $\beta s_4 = -0.1001$ $\beta sh_0 = 0.1009$ $\beta sh_1 = -0.0998$ $\beta sh_2 = 0.0448$ $\beta sh_3 = 0.0098$ $\beta sh_4 = -0.0190$	NaN NaN 18.9717 4.5597 0.2992 5.8149 3.4687 NaN 2.2944 0.1924 1.8421 3.2164	NaN NaN 0.9960 0.9830 0.7310 0.9860 0.9770 NaN 0.9650 0.8160 0.9960 0.9950	154.4407	177.3850
	Double Cox Gompertz	$Log_b = 1..6790$ $\beta s_0 = -6.3361$ $\beta s_1 = -0.8076$ $\beta s_2 = -0.2776$ $\beta s_3 = -1.1059$ $\beta s_4 = -0.3873$ $\beta sh_0 = -3.4759$ $\beta sh_1 = -0.1106$ $\beta sh_2 = 0.0623$ $\beta sh_3 = 0.0040$ $\beta sh_4 = -0.2713$	NaN 1.6709 1.2743 0.0523 0.9261 1.4118 NaN 0.2968 0.0053 0.1381 0.3170	NaN 0.0001*** 0.5262 1.1e-07*** 0.2324 0.7838 NaN 0.7093 $< 2e-16$*** 0.9768 0.3921	116.2852	137.2575
	Proposed Hybrid Coxph-Weibull	$\alpha = 1.7820$ $\gamma = 1.1310$ $\beta_1 = 0.0999$ $\beta_2 = -0.7003$ $\beta_3 = 0.1999$ $\beta_4 = -0.2003$ $\delta = -0.0027$ $p = 0.0001$	0.2658 0.0011 0.1035 0.2132 0.2101 0.1104 0.0078 0.0000	2e-11*** <2e-16*** 0.3343 0.0010** 0.3414 0.0696 0.7333 <2e-16***	114.9115	130.2077

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Interpretation: As the sample size increased to n = 50, the proposed hybrid again provided the lower AIC (114.91) compared to CoxPH (281.28), Weibull (116.70) and Double-Cox (154.44, 116.28). The value of AIC relative to sample size indicates improved stability and model fit. The α and γ parameters remained consistent, showing the Weibull component's steady contribution, while p continued to approach zero, implying the dominance of the Cox component in the mixture at this sample level.

Table 4.4 Simulation study on sample size 75

N	Model/Parameter	Coefficient	Standard Error	Prob	AIC	BIC
75	Coxph	$\beta_1 = 1.8945$ $\beta_2 = 0.6897$ $\beta_3 = 1.2407$ $\beta_4 = 0.8789$	0.1470 0.1311 0.1424 0.1358	1.38e-05*** 0.0024** 0.4107 0.0746*	479.2529	488.5228
	Weibull	$\beta_0 = 0.5000$ $\beta_1 = -0.5563$ $\beta_2 = 0.2930$ $\beta_3 = -0.1650$ $\beta_4 = 0.1190$ $\text{Log(Scale)} = -0.0961$ $\text{Scale} = 0.9990$	0.0113 0.1167 0.1119 0.1279 0.1181 0.0936	0.6536 1.9e-06 0.0088 0.1968 0.3138 0.3044	158.6041	172.5090
	Double-Cox Weibull	$\text{Log}_a = 0.2009$ $\text{Log}_b = -0..0170$ $\beta s_0 = -0.3741$ $\beta s_1 = -0.1813$ $\beta s_2 = -0.1504$ $\beta s_3 = -2.1480$ $\beta s_4 = -0.1001$ $\beta sh_0 = 1.7140$ $\beta sh_1 = 0.0219$ $\beta sh_2 = 0.0448$ $\beta sh_3 = -0.0303$ $\beta sh_4 = -0.2158$	2.037e-01 NaN 1.7950 0.8831 0.0628 1.28000 3.4687 NaN 0.1242 0.0092 0.1358 0.1458	0.3239 NaN 0.8349 0.8373 0.0167* 0.4218 0.0883 NaN 0,8600 4.64e-05*** 0.8232 0.1390	243.3236	271.1335
	Double Cox Gompertz	$\text{Log}_b = -0.0130.$ $\beta s_0 = 2.1190$ $\beta s_1 = -0.1791$ $\beta s_2 = -0.2224$ $\beta s_3 = -0.6667$ $\beta s_4 = -0.7287$ $\beta sh_0 = 1.2850$ $\beta sh_1 = 0.0377$ $\beta sh_2 = 0.0520$ $\beta sh_3 = -0.0087$ $\beta sh_4 = 0.1393$	5.9420 1.3740 0.7684 0.0426 0.7545 0.7470 5.9440 0.1121 0.0061 0.1187 0.1173	< 2e-16*** 0.1230 0.8160 1.75e-07*** 0.3770 0.3290 < 2e.16*** 0.7360 < 2e-16*** 0.9410 0.2350	142.1813	167.6737
	Proposed Hybrid Coxph-Weibull	$\alpha = 1.8160$ $\gamma = 1.1070$ $\beta_1 = 0.1000$ $\beta_2 = -0.6005$ $\beta_3 = 0.2001$ $\beta_4 = -0.2001$ $\delta = -0.0006$ $p = 0.0000$	0.1993 0.0000 0.0903 0.1635 0.1588 0.1010 0.0025 0.0000	< 2e-16*** 0.0000 0.2677 0.0002*** 0.2075 0.0476* 0.8208 < 2e-16***	143.2753	161.8152

Interpretation: the Double-Cox (142.18) shows a better fit compare to convoluted model (143.28), Weibull (158.60), and CoxPH (479.25) models. But the Double-Cox's

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parameters are unreliable. The hybrid model's mixture behavior became more evident as the sample size increased, maintaining flexibility between the Cox proportional component and the Weibull parametric component. The parameters (α = 1.8160, γ = 1.1070) reveal a moderately increasing hazard structure before a gradual decline, consistent with the shape of the empirical hazard curve.

Table 4.5 Simulation study on sample size 100

N	Model/Parameter	Coefficient	Standard Error	Prob	AIC	BIC
100	Coxph	$\beta_1 = 1.6692$ $\beta_2 = 0.5823$ $\beta_3 = 1.2458$ $\beta_4 = 0.6004$	0.1171 0.1274 0.1140 0.1206	21e-05*** 21e-05*** 0.0539 2.32e-05***	681.2456	691.6663
	Weibull	$\beta_0 = -0.0791$ $\beta_1 = -0.5401$ $\beta_2 = 0.5198$ $\beta_3 = -0.2381$ $\beta_4 = 0.5270$ Log(Scale)= 0.1139 Scale = 1.1200	0.1190 0.1164 0.1340 0.1252 0.1204 0.0790	0.5095 3.5e-06 0.0001 0.0572 1.2e-05 0.1492	174.5000	190.1310
	Double-Cox Weibull	$\text{Log}_a = 0.5993$ $\text{Log}_b = 0..4073$ $\beta s_0 = 0.0998$ $\beta s_1 = 0.0999.$ $\beta s_2 = -0.1057$ $\beta s_3 = 0.0999$ $\beta s_4 = -0.1001$ $\beta sh_0 = 0.1018$ $\beta sh_1 = -0.0989$ $\beta sh_2 = 0.0550$ $\beta sh_3 = 0.0101$ $\beta sh_4 = -0.0188$	0.0074 8.3583 0.3691 NaN 0.0376 1.4761 1.7294 8.4344 NaN 0.0067 0.6173 0.7388	< 2e-16*** 0.9611 0.7869 NaN 0.0050** 0.9461 0.9538 0.9538 NaN < 2e-16*** 0.9858 0.9797	276.1123	307.3744
	Double Cox Gompertz	$\text{Log}_b = -2.9030$ $\beta s_0 = 3.9390$ $\beta s_1 = -0.7574$ $\beta s_2 = -0.2141$ $\beta s_3 = 0.0501$ $\beta s_4 = -1.0240$ $\beta sh_0 = 2.8800$ $\beta sh_1 = 0.1098$ $\beta sh_2 = 0.0610$ $\beta sh_3 = 0.0696$ $\beta sh_4 = 0.1693$	8.4030 1.4350 0.6141 0.0372 0.7618 0.7734 8.4180 0.0958 0.0048 0.1385 0.1392	< 2e-16*** 0.0061*** 0.2174 9.04e-09*** 0.9475 0.1853 < 2e.16*** 0.2519 < 2e-16*** 0.6152 0.2241	193.9841	222.6410
	Proposed Convoluted Proportional Hazard	$\alpha = 1.8370$ $\gamma = 0.9273$ $\beta_1 = 0.1000$ $\beta_2 = -0.6001$ $\beta_3 = 0.2004$ $\beta_4 = -0.2000$ $\delta = 0.0360$ $p = 0.0085$	0.1455 0.0002 0.0770 0.1427 0.1369 0.0774 0.0000 0.0000	< 2e-16*** < 2e-16*** 0.1933 2.62e-05*** 0.1432 0.0098** 0.0000 < 2e-16**	53.8556	74.6970

Interpretation:**,** the convoluted model produced an unambiguously superior fit with the lowest AIC (53.86) and BIC (74.70), showing a sharp improvement over all competing models. The Double-Cox (AIC = 276.11, 193.98) although achieves a moderately low AIC but remains numerically unstable (NaN λ_2) and Weibull (AIC = 174.50) models performed less efficiently. The convergence of the shape (γ = 0.9273) and scale (α = 1.8370) parameters towards their theoretical expectations confirms the hybrid's efficiency in balancing short-term

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and long-term risk dynamics. The mixture weight (p = 0.0085) remains small, indicating that the Weibull part continues to dominate the long-term risk description, while the Cox component captures short term covariate proportional effects

Table 4.6 Simulation study on sample size 200

N	Model/Parameter	Coefficient	Standard Error	Prob	AIC	BIC
200	Coxph	$\beta_1 = 1.9123$	0.0847	00e-14***	1624.3160	1637.5090
		$\beta_2 = 0.5593$	0.0806	5.48e-13***		
		$\beta_3 = 1.2850$	0.0740	0.0007***		
		$\beta_4 = 0.9060$	0.0782	0.2066		
	Weibull	$\beta_0 = -0.0646$	0.0716	0.3665	385.6363	405.4262
		$\beta_1 = -0.5950$	0.0741	9.7e-16		
		$\beta_2 = 0.5414$	0.0697	7.8e-15		
		$\beta_3 = -0.2196$	0.0684	0.0013		
		$\beta_4 = 0.1015$	0.0741	0.1704		
		$\text{Log(Scale)} = -0.0445$	0.0549	0.4168		
		$\text{Scale} = 0.9560$				
	Double-Cox Weibull	$\text{Log}_a = 0.4931$	0.0428	< 2e-16***	573.7199	613.2997
		$\text{Log}_b = 3.8150$	5.3060	< 2e-16***		
		$\beta s_0 = 0.5239$	0.6283	0.4044		
		$\beta s_1 = 0.8013$	0.3943	0.0421		
		$\beta s_2 = -0.0801$	0.0251	0.0014**		
		$\beta s_3 = -0.0172$	0.3537	0.9612		
		$\beta s_4 = -0.2475$	0.3337	0.9612		
		$\beta sh_0 = -3.8140$	5.3080	< 2e-16***		
		$\beta sh_1 = -0.2706$	0.1113	0.0151		
		$\beta sh_2 = 0.0580$	0.0060	< 2e-16***		
		$\beta sh_3 = 0.2025$	0.1107	0.0675		
		$\beta sh_4 = -0.0025$	0.1078	0.9815		
	Double Cox Gompertz	$\text{Log}_b = 3.2620$	NaN	NaN	357.4512	393.7327
		$\beta s_0 = 2.1790$	0.3909	2.48e-08***		
		$\beta s_1 = -0.3505$	0.3917	0.3709		
		$\beta s_2 = -0.1914$	0.0122	< 2e-16***		
		$\beta s_3 = 1.0130$	0.3642	0.0054		
		$\beta s_4 = - - 0.6546$	0.3648	0.0728		
		$\beta sh_0 = -3.2770$	NaN	NaN		
		$\beta sh_1 = -0.0026$	0.0609	0.9656		
		$\beta sh_2 = 0.0490$	0.0017	< 2e-16***		
		$\beta sh_3 = -0.0635$	0.0556	0.2536		
		$\beta sh_4 = 0.0210$	0.0583	0.7182		
	Proposed Convoluted Proportional Hazard	$\alpha = 1.8341$	0.1035	< 2e-16***	97.4701	123.8567
		$\gamma = 0.9405$	0.0004	< 2e-16***		
		$\beta_1 = 0.1000$	0.0415	0.0159*		
		$\beta_2 = -0.9997$	0.1164	< 2e-05***		
		$\beta_3 = 0.3002$	0.0913	0.0010**		
		$\beta_4 = -0.1001$	0.0448	0.0253*		
		$\delta = -0.0004$	0.0015	0.7746		
		$p = 0.0000$	0.0000	0.0000		

Interpretation: For the largest simulated sample (n = 200), the proposed Convoluted model again achieved the best performance with AIC = 97.47 and BIC = 123.86, compared to CoxPH (AIC = 1624.32), Double-Cox (AIC = 573.71, 357.45), and Weibull (AIC = 385.64). The hybrid parameters ($\alpha = 1.8341$, $\gamma = 0.9405$, $p \approx 0$, $\delta \approx 0$) reflect model stability and convergence as sample size increases. These results indicate that the model retains efficiency and interpretability even with larger datasets,

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capturing both early risk escalation and the declining long-term hazard

4.6 Table Simulation Study on Sample Size 500

N	Model	Coefficient	Standard Error	Prob	AIC	BIC
500	Coxph	$\beta_1 = 1.6153$ $\beta_2 = 0.6207$ $\beta_3 = 1.2031$ $\beta_4 = 0.7216$	0.0505 0.0476 0.0467 0.0432	< 2e-16 < 2e-16 7.644-05 4.41e-14	4891.5710	4998.4300
	Weibull	$\beta_0 = 0.0320$ $\beta_1 = -0.4744$ $\beta_2 = 0.4828$ $\beta_3 = -0.1831$ $\beta_4 = 0.3310$ Log(Scale)= -0.0104 Scale $= 0.9900$	0.0468 0.0470 0.0433 0.0455 0.0404 0.0343	0.4900 < 2e-16 < 2e-16 5.7e-05 2.6e-16 0.7600	1036.5540	1061.8410
	Double-Cox Weibull	$\text{Log}_a = 0.6052$ $\text{Log}_b = 0.4121$ $\beta s_0 = 0.0100$ $\beta s_1 = 0.0998$ $\beta s_2 = -0.1084$ $\beta s_3 = -0.0999$ $\beta s_4 = -0.1002$ $\beta sh_0 = 0.1067$ $\beta sh_1 = -0.0946$ $\beta sh_2 = 0.0441$ $\beta sh_3 = 0.070$ $\beta sh_4 = -0.016$	0.0006 0.0060 0.7956 0.2661 0.0224 0.4495 0.2306 5.3080 0.0103 0.0078 0.0001 0.1008	< 2e-16*** < 2e-16*** 0.9000 0.7080 1.36e-06*** 0.8240 0.6640 < 2e-16*** 0.0051 1.65e-08*** < 2e-16*** 0.0051	1251.8219	1302.3972
	Double Cox Gompertz	$\text{Log}_b = 1.8620$ $\beta s_0 = 1.9940$ $\beta s_1 = 0.5163$ $\beta s_2 = -0.1898$ $\beta s_3 = 0.6388$ $\beta s_4 = -0.8325$ $\beta sh_0 = -1.8760$ $\beta sh_1 = -0.1054$ $\beta sh_2 = 0.0464$ $\beta sh_3 = 0.0672$ $\beta sh_4 = 0.0859$	2.9710 0.4429 0.2309 0.0115 0.2462 0.2357 2.9700 0.0397 0.0017 0.0400 0.0378	< 2e-16*** 6.72e-06*** 0.0253* < 2e-16*** 0.0095 0.0095 < 2e-16*** 0.0052 < 2e-16*** 0.0930 0.0228	994.4121	1040.7728
	Proposed Convoluted Proportional Hazard	$\alpha = 1.5629$ $\gamma = 1.8495$ $\beta_1 = 0.1000$ $\beta_2 = -1.0002$ $\beta_3 = 0.3000$ $\beta_4 = -0.1001$ $\delta = -0.0508$ $p = 0.0002$	0.8210 0.0246 0.0935 0.1938 0.1764 0.0899 0.0324 0.0000	0.0570 < 2e-16*** 0.2850 2.47e-07 0.0889 0.2653 0.1177 < 2e-16***	302.9532	336.6701

Interpretation: As the sample becomes larger (n=500), the convoluted proportional hazard model outperforms other models with lowest AIC and BIC (302.95 336.67), Weibull (1036.55, 1061.84), Cox PH (4891.57,4998.43) and Double Cox (1251.82, 1302.39). The shows consistency in performance of the new model

Table 4.7 Simulation Study on Sample Size 1000

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N	Model/Parameter	Coefficient	Standard Error	Prob	AIC	BIC
1000	Coxph	$\beta_1 = 1.6262$ $\beta_2 = 0.5809$ $\beta_3 = 1.1398$ $\beta_4 = 0.7333$	0.0353 0.0352 0.0321 0.0324	< 2e-16*** < 2e-16*** 4.56-05*** < 2e-16***	11425.4400	11445.0700
	Weibull	$\beta_0 = 0.0102$ $\beta_1 = -0.4778$ $\beta_2 = 0.5349$ $\beta_3 = -0.1321$ $\beta_4 = 0.3029$ Log(Scale)= -0.0321 Scale $= 0.9680$	0.0323 0.0312 0.0302 0.0306 0.0302 0.0248	0.7500 < 2e-16 < 2e-16 1.6e-05 < 2.6e-16 0.2000	2034.6560	2064.1030
	Double-Cox Weibull	$\mathrm{Log}_a = 0.4244$ $\mathrm{Log}_b = 1.5424$ $\beta s_0 = 0.2073$ $\beta s_1 = 0.3303$ $\beta s_2 = -0.0938$ $\beta s_3 = 0.7762$ $\beta s_4 = -0.2388$ $\beta sh_0 = -1.2100$ $\beta sh_1 = -1.1187$ $\beta sh_2 = 0.0523$ $\beta sh_3 = 0.0122$ $\beta sh_4 = -0.0532$	0.0212 3.7520 0.3254 0.1746 0.0108 0.1770 0.1716 3.7530 0.0440 0.0023 0.0458 0.0443	< 2e-16*** < 2e-16*** 0.5241 0.0585 < 2e-16*** 1.16e-05*** 0.1638 < 2e-16*** 0.0070 < 2e-16*** 0.7898 0.2294	2525.7838	2584.6769
1000	Double Cox Gompertz	$\mathrm{Log}_b = -7.0610$ $\beta s_0 = 0.8692$ $\beta s_1 = 0.2813$ $\beta s_2 = -0.1629$ $\beta s_3 = 0.2745$ $\beta s_4 = -0.0619$ $\beta sh_0 = 7.7047$ $\beta sh_1 = -0.0757$ $\beta sh_2 = 0.0455$ $\beta sh_3 = 0.1494$ $\beta sh_4 = 0.0493$	0.0091 0.3054 0.1821 0.0084 0.1843 0.1843 0.0021 0.0296 0.0013 0.0308 0.0307	< 2e-16*** 0.0044** 0.1225* < 2e-16*** 0.1365 0.7370 < 2e-16*** 0.0106 <2e-16*** 1..25e-06*** 0.1082	1957.6031	2011.5884
	Proposed Convoluted Proportional Hazard	$\alpha = 1.5830$ $\gamma = 1.7970$ $\beta_1 = 0.0999$ $\beta_2 = -1.0000$ $\beta_3 = 0.3003$ $\beta_4 = -0.0999$ $\delta = -0.0724$ $p = 0.0005$	0.6090 0.0003 0.0589 0.1282 0.1102 0.0572 0.0572 0.0000	0.0093** < 2e-16*** 0.0892 6.04e-15*** 0.0065** 0.0807 0.0311 < 2e-16***	621.8292	638.6876

Interpretation: As sample size increased further (n=1000), the convoluted proportional hazard model outperforms other models with lowest AIC and BIC (621.82 638.68), Weibull (2034.65, 2064.10), Cox PH (11425.44,11445.07) and Double Cox (2525.78, 2584.67).The convoluted proportional hazard model demonstrated superior power over other existing model showing better goodness of fit through the least values in AIC and BIC.

Table 4.7 Analysis on real life data of sample size 257

Model	MLE	Standard Error	p-value	- Loglik	AIC	BIC
	$\alpha = 1.5262$ $\gamma = 0.8704$	0.0245 0.7417	0.0000 0.2406	-4233.682	8487.3635	8522.8543

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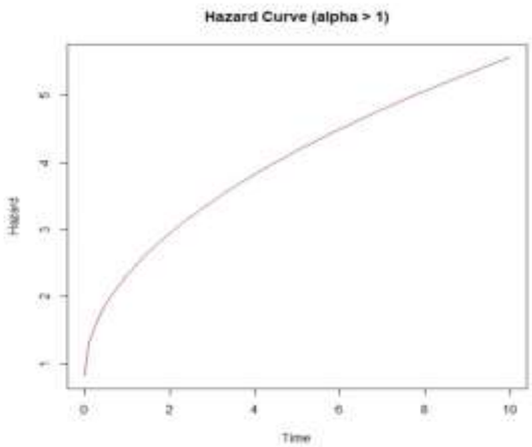
Proposed Convoluted proportional hazard model	$\beta_1 Mart = 1.3563$ $\beta_2 Age = -0.0426$ $\beta_3 Dis = 0.0041$ $\beta_4 Off = -1.0140$ $\beta_5 Edu = 0.0919$ $\beta_6 Empl = -1.2406$ $p = 0.9681$ $\delta = 1.0000$	0.5322 0.0304 0.0528 0.2509 0.0636 0.1766 0.0412 0.0000	0.0095 0.1621 0.9383 0.0001 0.1484 0.0000 0.0000 0.0000			
Weibull	$\alpha = 1.5041$ $\gamma = 3.2862$ $\beta_1 Mart = 0.1162$ $\beta_2 Age = -0.0188$ $\beta_3 Dis = 0.0138$ $\beta_4 Off = -0.1094$ $\beta_5 Edu = -0.0296$ $\beta_6 Empl = -0.3123$	0.0237 0.3462 0.0535 0.0034 0.0058 0.0184 0.0116 0.0501	0.0000 0.0000 0.0183 0.0006 0.0176 0.0000 0.0107 0.0000	-4251.8100	8523.6195	8559.1103
Double-Cox	$\lambda_1 = 0.5946$ $\lambda_2 = 0.0944$ $\beta_1 Mart = -0.2267$ $\beta_2 Age = 0.0369$ $\beta_3 Dis = 0.0935$ $\beta_4 Off = -0.0448$ $\beta_5 Edu = -0.0055$ $\beta_6 Empl = -0.0329$ $p = 0.0245$	0.0000 0.0095 0.0567 0.0041 0.0079 0.0145 0.0109 0.0443 NaN	0.0000 0.0000 0.0198 0.0000 0.0000 0.0844 0.9972 0.8710 NaN	-4329.6470	8699.2940	8770.2750
CoxPH	$\beta_1 Mart = 0.0957$ $\beta_2 Age = -0.0154$ $\beta_3 Dis = 0.0294$ $\beta_4 Off = -0.1017$ $\beta_5 Edu = -0.0216$ $\beta_6 Empl = -0.3087$	0.0501 0.0026 0.0053 0.0175 0.0107 0.0381	0.0562 0.0000 0.0000 0.0000 0.0437 0.0000	-4528.9640	9077.9282	9113.4190

Interpretation: From the real life data set, the proposed convoluted model once again outperformed all competing models. The model yielded the smallest AIC (8487.36) and the highest log-likelihood (-4233.68), compared to the Weibull (AIC = 8523.62), Double-Cox (AIC = 8699.29), and CoxPH (AIC = 9077.93). Significant parameters included marital status ($\beta_1 = 1.3563$, $p < 0.01$), employment ($\beta_6 = -1.2406$, $p < 0.001$), and offense type ($\beta_4 = -1.0140$, $p < 0.001$), confirming that being married and employed reduce recidivism risk, while certain offense types elevate it. The mixture parameter $p = 0.9681$ and $\delta = 1.0000$ suggest that both components (Cox and Weibull) jointly influence the hazard structure. The shape ($\gamma = 0.8704$) and scale ($\alpha = 1.5262$) parameters confirm a non-monotonic hazard consistent with criminological patterns—high short-term risk followed by gradual stabilization among long-term survivors



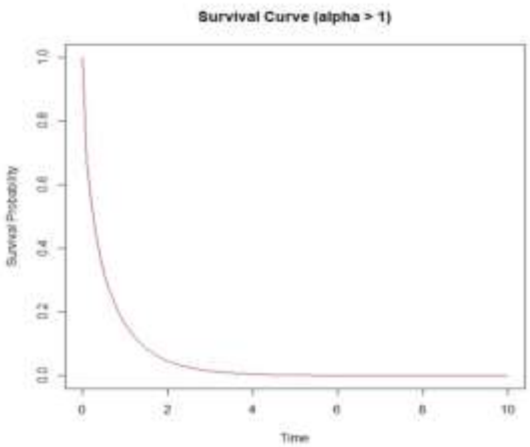


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Figure 4.1 Hazard and Survival Curve of the Proposed Model for $\alpha > 1$

Interpretation: The plots presented above illustrate the behavior of the proposed Convoluted CoxPH–Weibull model under the condition α > 1. The hazard curve reveals a monotonically increasing trend, signifying that the instantaneous risk of event occurrence intensifies progressively with time. This indicates that individuals or subjects with longer exposure durations experience cumulative vulnerability, consistent with real-life phenomena such as long-term recidivism where the risk of reoffending increases with continued exposure to underlying criminogenic factors.The observed patterns for α > 1 therefore affirm that the proposed model effectively accommodates increasing hazard dynamics and provides a flexible structure for analyzing both short-term deterrence and long-term relapse tendencies. The interplay between the Cox and Weibull components enhances the model's explanatory power, allowing it to reflect accelerating risk behavior that would be inadequately captured by a single-component model.

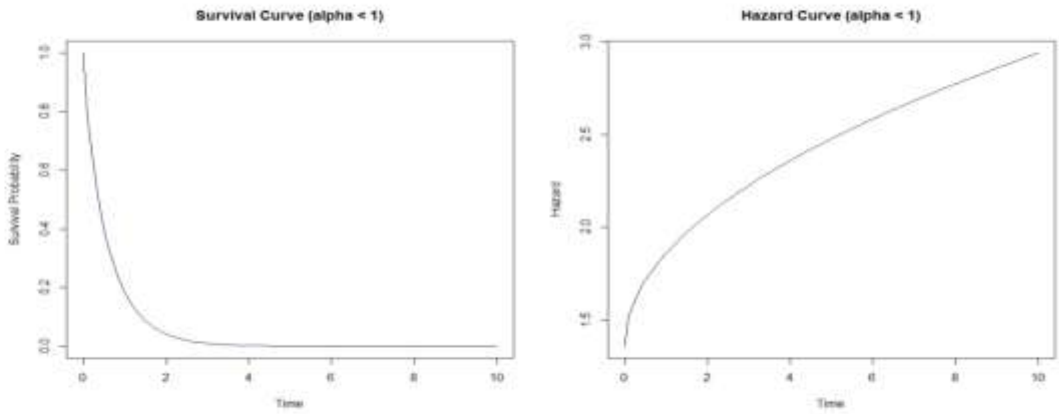

Figure 4.2 Hazard and Survival Curve of the Proposed Model for $\alpha < 1$
Interpretation:

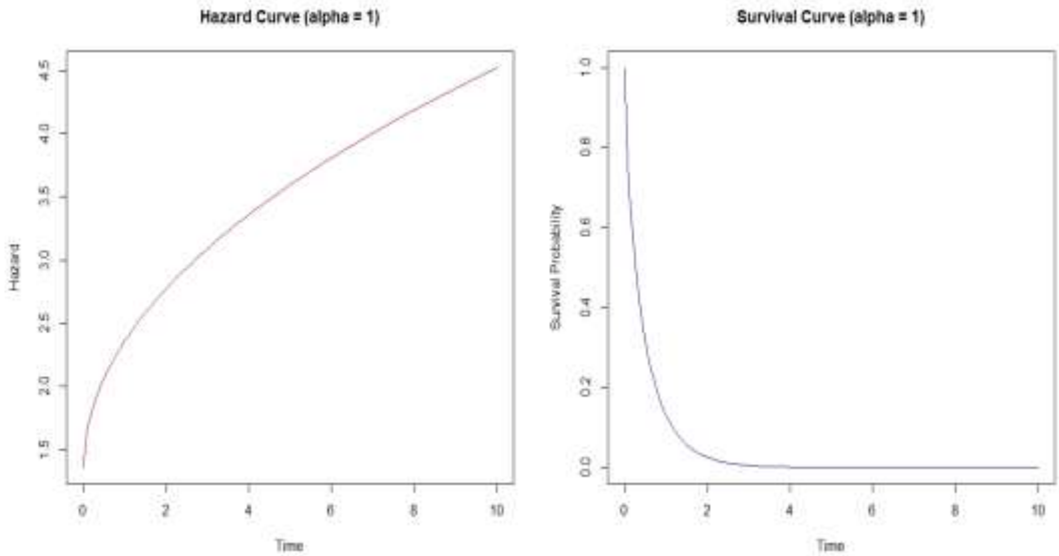

Figure 4.3: The Hazard and Survival Curve of the Proposed Model for $\alpha = 1$
Interpretation: When alpha is equals one, it implies that the risk (hazard) is constant.

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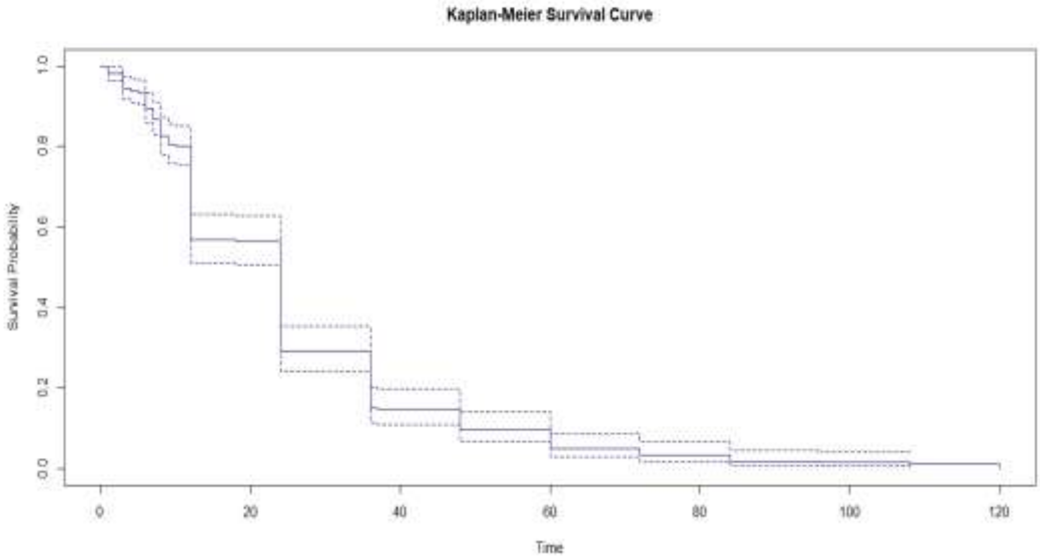

Figure 4.4: The Kaplan Meier Survival Curve of the Proposed Model

Interpretation: figure 4.4 shows a steady decline in survival probability over time, with sharp drops during the early period. This implies that most recidivism events occur shortly after release, while the remaining population exhibits a slower decline, suggesting the existence of a long-term desisting group.

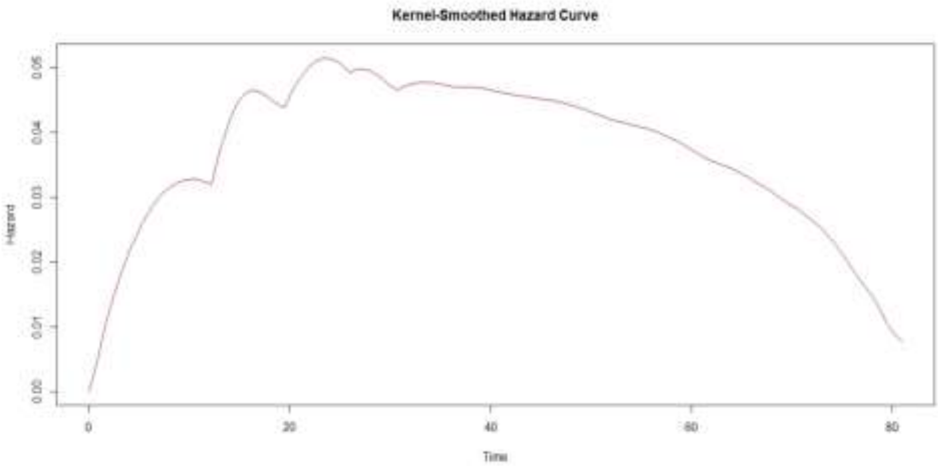

Figure 4.5: The Kernel-Smoothed Hazard Curve of the Proposed Model

Interpretation: figure 4.5 shows an early rise in hazard rates, peaking around the 20th time unit, and a gradual decline thereafter. This pattern indicates that the risk of reoffending is highest in the early months after release and decreases over time—a dynamic perfectly captured by the proposed hybrid model's convolution of short- and long-term components.

5.0 Conclusion

The result of the analysis leads to the conclusion that there is a high prevalence rate of recidivism in Ekiti State since approximately 32% of inmates recidivated. Of all the risk factors considered using the new developed convoluted proportional hazard model, it is clear that that marital status, offence category and employment status are the only statistically significant covariates at 0.05 level of significance. The result also demonstrates that the proposed Convoluted Proportional Hazard Model outperforms the traditional Cox, Weibull models across both simulated and real-life datasets and generally, it shows strength and better performance on

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Double-Cox, and Its strength lies in its ability to accommodate heterogeneity in risk timing capturing both immediate post-release risks (via the Cox component) and gradual long-term desistance (via the Weibull component).

The consistent superiority in AIC and BIC values across varying sample sizes underscores the hybrid model's stability, robustness, and scalability. While the Double-Cox model captures dependence through shared frailty, it lacks the parametric flexibility of the Weibull component. The proposedCox model addresses this gap by incorporating the Weibull's parametric control and the Cox model's proportional hazards framework within a single coherent system.

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