Study of Finite Element Method for First Order Linear Differential Equations

Vikas Poply
Associate Professor, Department of Mathematics,
Kurukshetra University, Kurukshetra

Savita
PG Student, Department of Mathematics,
Kurukshetra University, Kurukshetra

Abstract:
This research focuses on the application of the Finite Element Method (FEM) for obtaining numerical solutions to linear differential equations that arise in various scientific and engineering problems. The method works by discretizing the problem domain into smaller subdomains called elements, which are connected at nodes. Unknown variables, such as displacement or temperature, are approximated at these nodes using suitable shape functions. A weak formulation of the governing equations is developed, and element-level equations are assembled into a global system. Appropriate boundary conditions are incorporated to represent real physical situations. The resulting system is then solved using numerical techniques to obtain approximate solutions. FEM is widely used in structural analysis, mechanical design, heat transfer, fluid flow, and biomedical engineering The results obtained from FEM provide reliable approximations that support better design, improve safety, and reduce dependence on experimental testing. This study highlights the effectiveness and versatility of FEM as a powerful computational tool for solving real-world engineering problems involving differential equations.

Keywords:
Finite Element Method, Numerical Approximation, Differential Equations

1. Introduction

Differential equations are widely used to model physical phenomena such as heat transfer, fluid flow, structural deformation, and electromagnetic fields. Many real-world engineering problems involve complex geometries and boundary conditions, making analytical solutions difficult or even impossible. To overcome these limitations, the Finite Element Method (FEM) was developed as a powerful numerical technique for obtaining approximate solutions of ordinary and partial differential equations by dividing the problem domain into smaller, simpler subdomains known as finite elements.

The origin of FEM can be traced back to 1943, when Courant [1] used piecewise polynomial interpolation over triangular regions to solve torsion problems, thereby laying the mathematical foundation of the method. During the 1950s, FEM began to develop rapidly for solving complex structural problems,particularly in aerospace enginee--ring. In 1956, Turner, Clough, Martin, and Topp formally applied FEM to structural analysis for the first time, demonstrating its practical usefulness in engineering applications.FEM beyond structural mechanics to areas such as heat transfer, fluid mechanics, and solid mechanics, establishing it as a general numerical technique. From the 1970s onwards, with the rapid advancement of digital computers, FEM evolved into a highly powerful and widely used method capable of solving linear and nonlinear, static and dynamic engineering problems.

The original formulation of "Finite Element Method" was introduced in 1990 by Clough [2], who made significant contributions to its development and popularization and is often referred to as the Father of the Finite Element Method.

DOI: https://doi.org/10.5281/zenodo.20784769

ISSN NO-2584-2706

Figure 1. Applications of FEM

The Finite Element Method (FEM) is a numerical approach used to analyse and solve engineering and physical problems that are described by differential equations. It is especially helpful for problems involving complex shapes, varying material properties,[3] and complicated boundary conditions, where exact analytical solutions are not possible. Applications of FEM has been shown in Figure 1.

The Finite Element Method was developed to overcome the limitations of classical analytical and numerical methods when solving complex engineering problem. Classical methods cannot easily handle structures with irregular shapes and boundaries. Engineering problems often involve mixed and complex boundary conditions. For solving such type of problems, FEM plays a major role and his become the motivation of current study which can be displayed in Figure 2.

Figure 2. Motivation of FEM

2. Finite element method

The Finite Element Method (FEM) is a numerical method to solve the complex physical problems to find the approximation solutions, mainly those that are controlled by partial differential equation It breaks the complex, hard problem into easier, clear, small, simple parts called "finite elements." Each element is solved separately [4], and then all the elements are recombined to give an approximate solution for the whole problem. Methodology of FEM has been shown in Figure 3.

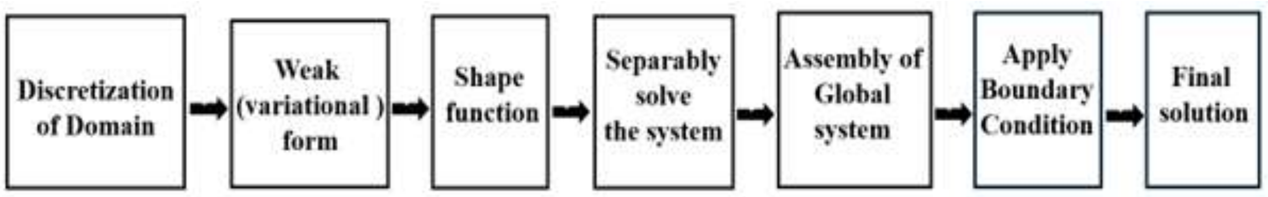

Figure 3. Methodology of FEM

3. Numerical procedure for Solving First-Order Differential Equation

For first order differential equation in space, hence requires one boundary condition.

Consider a general 1st order ODE on a domain $\Omega = (a, b)$:

$$\frac{du}{dx} + p(x)u = f(x), \qquad a < x < b$$

with a boundary condition

$$u(a) = u_a$$

Divide the interval $[a, b]$ into n finite elements:

$$a = x_0 < x_1 < x_2 - -- < x_n = b$$

On each element, approximate the solution using a linear shape function (first order FEM)

$$u(x) \approx u_h(x) = \sum_{i=1}^{n} U_i \, N_i(x)$$

Multiply the differential equation by a test function $v(x)$ and integrate over the domain:

$$\int_a^b \left(\frac{du}{dx} + p(x)u\right) v \, dx = \int_a^b f(x) v \, dx$$

Now integrate the first term by parts:

$$\int_a^b \frac{du}{dx} v \, dx = [uv]_a^b - \int_a^b u \frac{dv}{dx} \, dx$$

DOI: https://doi.org/10.5281/zenodo.20784769

Substitute back:

$$-\int_a^b u\frac{dv}{dx}\,dx+\int_a^b p(x)uv\,dx+[uv]_a^b = \int_a^b fv\,dx$$

$uv = N_iN_j = 0$ unless nodes i and j belong to the same element.
In Galerkin FEM, choose
$v = N_j(x)$
Substitute into weak form:

$$\sum_i U_i\left[-\int_a^b N_i\frac{dN_j}{dx}\,dx+\int_a^b p\,(x)N_iN_j\,dx\right] = \int_a^b f\,N_j\,dx$$

On a typical element $e = [x_k, x_{k+1}]$, define
Element Stiffness Matrix

$$k_{ij}^{(e)} = -\int_{x_k}^{x_{k+1}} N_i\frac{dN_j}{dx}\,dx + \int_{x_k}^{x_{k+1}} p(x)\,N_iN_j\,dx$$

Element load vector

$$f_j^{(e)} = \int_{x_k}^{x_{k+1}} f(x)\,N_j\,dx$$

Assemble all element matrices into the global system
$KU = F$
Apply boundary condition (e.g. $u(a) = u_a$) by modifying the system.
Solve for nodal values U, then

$$u_h(x) = \sum_i U_i\,N_i(x)$$

This gives the finite element approximation of the first order ODE.
Let us take a particular example
$\frac{du}{dx}+u = x$ on $0\le x\le 1$
Boundary condition:
$u(0) = 0$

Exact solution of this problem:

$$\frac{du}{dx}+u = x$$

$C.F. = C_1e^{-x}$
C_1 is constant.

$$P.I. = \frac{x}{D+1} = (D+1)^{-1}(x) = (1-D+D^2-\cdots)(x) = x-1$$

Complete solution is: $u = C_1e^{-x}+x-1$
Using boundary condition:
$u(0) = C_1e^0+0-1 = C_1-1 = 0$
$C_1 = 1$
$u = e^{-x}+x-1$
Now, this problem solved by finite element method:

Step 1: Weak form (variational form):
Multiply by weight function w(x) and integrate over the domain:

$$\int_0^1 w\left(\frac{du}{dx}+u-x\right)dx = 0$$

Using GalerKin's Method, choose
$w = N_i$(same as shape function)

So,

$$\int_0^1 N_i\frac{du}{dx}dx+\int_0^1 N_i\,u\,dx = \int_0^1 N_i\,x\,dx(1)$$

Step 2: Discretization of the domain:
Divide $[0,1]$ into 5 linear elements (for clarity)
Nodes: $x_1 = 0, x_2 = 0.2,\ x_3 = 0.4,\ x_4 = 0.6,\ x_5 = 0.8,\ x_6 = 1$
Element length: $h = 0.2$

Step 3: Shape function (linear element)
For each element, $N_1 = \frac{x_2-x}{h}, N_2 = \frac{x-x_1}{h}$
Approximate solution: $u(x) = N_1u_1(x)+N_2u_2(x)$

Step 4: Element equation – substitute into weak form over one element $[x_1, x_2]$
Term 1: (Derivative form)

$$\int_{x_1}^{x_2} N_i\frac{du}{dx}\,dx$$

where $u(x)$ = unknown solution, $\frac{du}{dx}$ = gradient (convection term), N_i = weight function
In FEM, we approximate u inside an element as: $u(x) = N_1u_1+N_2u_2$
Term 1 becomes

$$\int_{x_1}^{x_2} N_i\frac{du}{dx}dx = \int_{x_1}^{x_2} N_i\left(\frac{dN_1}{dx}u_1+\frac{dN_2}{dx}u_2\right)dx$$

DOI: https://doi.org/10.5281/zenodo.20784769

Split it: $\int_0^1 N_i \frac{du}{dx}\,dx = u_1 \int_0^1 N_i \frac{dN_1}{dx}\,dx + u_2 \int_0^1 N_i \frac{dN_2}{dx}\,dx$

Take one element $[x_1, x_2]$, length $h = 0.2$

Shape functions: $N_1 = \frac{x_2 - x}{h}, N_2 = \frac{x - x_1}{h}$

Compute the integrals

(a) $\int_{x_1}^{x_2} N_1 \frac{dN_1}{dx} dx = \int_{x_1}^{x_2} N_1 (-5) dx = -5 \int_{x_1}^{x_2} N_1\,dx = -0.5$

(b) $\int_{x_1}^{x_2} N_1 \frac{dN_2}{dx} dx = 5 \int_{x_1}^{x_2} \frac{x_2 - x}{h} dx = \frac{5}{h} \int_{x_1}^{x_2} (x_2 - x) dx = \frac{5}{h}\left[\frac{h^2}{2}\right] = \frac{5h}{2} = 0.5$

(c) $\int_{x_1}^{x_2} N_2 \frac{dN_1}{dx} dx = -\frac{5}{h}\int_{x_1}^{x_2} (x - x_1) dx = -0.5$

(d) $\int_{x_1}^{x_2} N_2 \frac{dN_2}{dx}\,dx = \frac{5}{h}\int_{x_1}^{x_2} (x - x_1)\,dx = 0.5$

Finally, $\int N_i \frac{dN_j}{dx}\,dx = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$

Term 2 – Reaction term

$$\int N_1^2\,dx = \int_{x_1}^{x_2} \left(\frac{x_2 - x}{h}\right)^2 dx = \frac{h}{3}$$

Next Term $\int N_1 N_2\,dx = \frac{1}{h^2}\int_{x_1}^{x_2} (x_2 - x)(x - x_1)\,dx = \frac{1}{h^2}\int_{x_1}^{x_2} (x_2 x - x_2 x_1 - x^2 + x x_1)\,dx$

$$= \frac{1}{6h^2}(x_2 - x_1)^3 = \frac{h^3}{6h^2} = \frac{h}{6}$$

Finally, $\int N_i N_j\,dx = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Substitute $h = 0.2$ then $\int N_i N_j\,dx = \begin{bmatrix} 0.066667 & 0.033333 \\ 0.033333 & 0.066667 \end{bmatrix}$

Term 3: Load Vector

Element Load Vector (For All Elements)

$\int N_i\,x\,dx$

Evaluate for $i = 1,2$

For N_1:

$$\int_{x_1}^{x_2} N_1\;x\,dx = \int_{x_1}^{x_2} \frac{x_2 - x}{h}\,x\,dx$$

$$= \frac{1}{h}\left[\frac{x_2 x^2}{2} - \frac{x^3}{3}\right]_{x_1}^{x_2} = \frac{h}{6}(x_1 + 2x_2)$$

For N_2:

$$\int_{x_1}^{x_2} N_2\;x\,dx = \int_{x_1}^{x_2} \left(\frac{x - x_1}{h}\right) x\,dx$$

$$= \frac{1}{h}\int_{x_1}^{x_2} (x^2 - x_1 x)\,dx = \frac{h}{6}(x_1 + 2x_2)$$

Final Load Vector (for element) $\frac{h}{6}\begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$

Step 5: Final Element Stiffness Matrix

$$K^e = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.066667 & 0.033333 \\ 0.033333 & 0.066667 \end{bmatrix}$$

$$= \begin{bmatrix} -0.433333 & 0.533333 \\ -0.466667 & 0.566667 \end{bmatrix}$$

Element Load Vector $F^e = \frac{h}{6}\begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$

Element 1 [0, 0.2] $F^e = \frac{0.2}{6}\begin{bmatrix} 2(0) + 0.2 \\ 2(0.2) + 0 \end{bmatrix} = 0.033333\begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.006667 \\ 0.013333 \end{bmatrix}$

Node	Element	x_1	x_2	F_1	F_2
1–2	1	0.0	0.2	0.006667	0.013333
2–3	2	0.2	0.4	0.020000	0.026667
3–4	3	0.4	0.6	0.033333	0.040000
4–5	4	0.6	0.8	0.046667	0.053333
5–6	5	0.8	1.0	0.060000	0.066667

DOI: https://doi.org/10.5281/zenodo.20784769

Step 6: Global Assembly (Numerically)
Assemble all element matrices into global system:
$KU = F$

Global Stiffness Matrix K

$$K = \begin{bmatrix} -0.433333 & 0.533333 & 0 & 0 & 0 & 0 \\ -0.466667 & 0.133334 & 0.533333 & 0 & 0 & 0 \\ 0 & -0.466667 & 0.133334 & 0.533333 & 0 & 0 \\ 0 & 0 & -0.466667 & 0.133334 & 0.533333 & 0 \\ 0 & 0 & 0 & -0.466667 & 0.133334 & 0.533333 \\ 0 & 0 & 0 & 0 & -0.466667 & -0.466667 \end{bmatrix}$$

Manually Assembly of Load Vector
Add them node wise

$$F^1 = 0.006667$$
$$F^2 = 0.013333 + 0.020000 = 0.033333$$
$$F^3 = 0.026667 + 0.033333 = 0.060000$$
$$F^4 = 0.040000 + 0.046667 = 0.086667$$
$$F^5 = 0.053333 + 0.060000 = 0.113333$$
$$F^6 = 0.066667$$

Apply Boundary Condition

$u_1 = 0$
Remove row 1 & column 1

Final System

$$\begin{bmatrix} 0.133334 & 0.533333 & 0 & 0 & 0 \\ -0.466667 & 0.133334 & 0.533333 & 0 & 0 \\ 0 & -0.466667 & 0.133334 & 0.533333 & 0 \\ 0 & 0 & -0.466667 & 0.133334 & 0.533333 \\ 0 & 0 & 0 & -0.466667 & -0.466667 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0.033333 \\ 0.060000 \\ 0.086667 \\ 0.113333 \\ 0.066667 \end{bmatrix}$$

Table 1. Comparison of exact solution with FEM solution

u	Exact solution	FEM solution
0	0.0000	0
0.2	0.0182	0.02038
0.4	0.0703	0.06991
0.6	0.1488	0.15036
0.8	0.2493	0.24858
1	0.3678	0.36942

Table 1 shows the comparison of the FEM solution with exact solution for first order differential equations

4.Conclusions

Finite Element Method (FEM) solution is in very close agreement with the exact solution at all selected nodal points. The numerical values

DOI: https://doi.org/10.5281/zenodo.20784769

obtained through FEM show only minor deviations from the exact values, indicating high accuracy and reliability of the method for solving the given first-order differential equation. As the values of u increase from 0 to 1, the FEM results consistently follow the exact solution trend with negligible error. This comparison confirms that FEM is an effective and precise numerical technique for approximating solutions to first-order differential equations.

4. References

[1] R. L. Courant, "Variational Methods for the Solution of Problems of Equilibrium and Vibration," Bulletin of the American Mathematical Society, 49, 1943, 1-23.

[2] R. W, Clough, “Original formulation of the finite element method,” Finite Elements in Analysis and Design, 7(2), 1990, 89-101.

[3] S.S. Shastry,” Introductory Methods of Numerical Analysis, Fifth Edition,” PHI learning private limited, 2012.

[4] J. Stoer and R. Bulirsch: Introduction to Numerical Analysis, Springer International Edition, 1976.

[5] J. N. Reddy: An Introduction to the Finite Element Method, Second Edition, McGraw-Hill, Inc. 1993.

DOI: https://doi.org/10.5281/zenodo.20784769