

The Gbenga Gideon Integral Transform: A Four-Parameter Unifying Framework for Laplace-Type Transforms, with Rigorous Theory, Engineering Applications, and Numerical Validation

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Abstract

Over the past three decades, a large family of Laplace-type integral transforms has been proposed, including the Sumudu, Elzaki, Aboodh, Mohand, Mahgoub, Kamal, Shehu, Sadik, Sawi, and Natural transforms, each defined by a kernel of the form $\varphi(v) e^{-\psi(v)t}$ and each accompanied by its own operational calculus. The proliferation of these transforms has produced a fragmented literature in which essentially equivalent operational results are repeatedly rederived for each new kernel.

This study presents the Gbenga Gideon (G.G.) integral transform, a four-parameter generalized integral transform defined by $\mathcal{G}\{f(t)\}(v) = p v^m \int_0^\infty e^{-qv^wt} f(t) dt$, and establishes it as a single unifying framework from which the operational calculus of at least thirty-eight published transforms is recovered by parameter specialization.

The transform is developed axiomatically. Existence and convergence are established for piecewise continuous functions of exponential order, and the fundamental operational properties, linearity, shifting, change of scale, transforms of derivatives, multiplication by powers of t , convolution, inversion, and uniqueness, are proved as formal theorems. The framework is then applied to canonical initial value problems in electrical circuit analysis, heat conduction and cooling, population growth, radioactive decay, and mechanical vibrations.

Nine theorems with complete proofs are presented, together with a parameter dictionary mapping the G.G. integral transform onto existing transforms. Analytical solutions obtained through the G.G. integral transform is validated against high-accuracy Runge-Kutta integration: across eight benchmark problems the maximum absolute discrepancy does not exceed 2.3×10^{-7} and root-mean-square errors lie between 10^{-11} and 10^{-8} , confirming exactness to numerical precision. A symbolic benchmark shows the transform-domain route resolving a damped oscillator initial value problem approximately 3.4 times faster than direct symbolic integration.

The G.G. integral transform provides a mathematically rigorous, computationally efficient, and pedagogically economical umbrella for the Laplace-type transform family: a property proved once for the G.G. integral transform holds automatically for every member obtained by specialization.

Keywords:

Generalized Integral Transform; G.G. Integral Transform; Laplace-Type Transforms; Operational Calculus; Ordinary Differential Equations; Electrical Circuits; Mechanical Vibrations; Convergence Analysis; Unification.

1. Introduction

Integral transforms are among the most powerful tools of applied mathematics. By mapping a differential or integral problem posed in a physical domain (usually time) into an algebraic problem posed in a transform domain, they convert the analytic difficulty of solving differential equations into

the comparatively routine work of algebraic manipulation and inversion. Since Laplace formalized the exponential-kernel transform that bears his name, this strategy has become the standard solution technique for linear initial value problems arising in electrical circuit theory, heat conduction, population dynamics, radioactive decay, control engineering, and mechanical vibrations [1–6].

The classical Laplace transform,

$$\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

is not, however, the only exponential-kernel transform in use. Beginning with Watugala's Sumudu transform in 1993 [7], the literature has seen a sustained proliferation of “new” integral transforms, each obtained by attaching a prefactor $\varphi(v)$ to the integral, rescaling the exponent, or both: the Elzaki [9], Aboodh [11], Mahgoub [12], Mohand [13], Kamal [14], Sawi [15], Shehu [16], Sadik [17], Natural [18], Tarig, Pourreza, SEE, Rishi, Soham, and many other transforms. Each of these papers rederives, for its particular kernel, the same suite of operational properties: linearity, transforms of elementary functions, shifting theorems, derivative rules, and applications to the same canonical differential equations.

This study takes the opposite approach. Rather than adding one more isolated kernel to the list, we introduce a transform with four free parameters chosen so that the entire exponential-kernel family becomes a single object of study. We define the G.G. (Gbenga Gideon) integral transform of a function $f(t)$, $t \geq 0$, by

$$G(v) = \int_0^{\infty} p v^m e^{-qv^w t} f(t) dt, \quad (1.1)$$

where $p > 0$ and $q > 0$ are positive real constants and m, w are real exponents with $w \neq 0$. The kernel is $k(t, v) = p v^m e^{-qv^w t}$. Choosing $(p, q, m, w) = (1, 1, 0, 1)$ recovers the Laplace transform; $(1, 1, -1, -1)$ recovers the Sumudu transform; $(1, 1, 1, -1)$ recovers the Elzaki transform; $(1, 1, -1, 1)$ recovers the Aboodh transform; and so on through at least thirty-eight published transforms, catalogued in Section 10.

The unification is not merely notational. Because every operational property of the G.G. integral transform is proved once, in Sections 7 to 9, for arbitrary admissible (p, q, m, w) , every specialization inherits those properties without further proof. A derivative rule, a convolution theorem, or a uniqueness result established for the G.G. integral transform is simultaneously a derivative rule, convolution theorem, and uniqueness result for the Laplace, Sumudu, Elzaki, Aboodh, Mohand, Mahgoub, Kamal, Shehu, Sadik, and Natural transforms. The framework therefore compresses a fragmented literature into one theory, in the spirit of Jafari's recent general transform [19] but with an explicitly parameterized power-law structure $p v^m$ and $q v^w$ that makes the recovery of each published transform a matter of reading four values from a dictionary.

The remainder of the paper is organized as follows. Section 2 reviews the classical and modern transform literature. Section 3 states the research gap and Section 4 the contributions. Section 5 collects mathematical preliminaries. Section 6 defines the transform and its admissible parameter set. Section 7 establishes existence, convergence, and asymptotic behavior. Section 8 develops the fundamental operational properties. Section 9 presents the inversion, uniqueness, and convolution theorems. Section 10 develops the unification dictionary and a comparative study. Section 11 applies the transform to engineering problems with full physical discussion. Sections 12 and 13 report numerical validation and error analysis, Section 14 analyses computational complexity, and Sections 15 to 18 present discussion, limitations, conclusions, and future work.

2. Literature Review

2.1 Classical integral transforms

The systematic theory of the Laplace transform was consolidated by Doetsch [2] and Widder [3], who established the modern treatment of existence for functions of exponential order, the complex (Bromwich) inversion integral, and Lerch's uniqueness theorem. Standard engineering treatments [1,

4, 5] cemented the transform as the default method for linear constant-coefficient initial value problems. Alongside it, the Fourier transform handles problems on the whole line and spectral analysis [6], the Mellin transform handles scale-invariant and asymptotic problems, and the Hankel transform handles axisymmetric problems in cylindrical geometry [1]. These four classical transforms differ fundamentally in kernel structure (e^{-st} , $e^{-i\omega t}$, t^{s-1} , $tJ_\nu(st)$ respectively) and are genuinely distinct mathematical objects.

2.2 The modern family of Laplace-type transforms

The modern generalized transforms are different in character: all of them retain an exponential kernel in t and differ only in how the transform variable enters. The principal members are summarized below; their kernels are tabulated in Table 1 (Section 10).

Sumudu transform. Watugala [7] introduced $S\{f\}(v) = \frac{1}{v} \int_0^\infty e^{-t/v} f(t) dt$ for control engineering problems, emphasizing its units-preserving property: the Sumudu image of a function has the same dimensions as the function itself. Weerakoon [8] extended it to partial differential equations, and Belgacem, Karaballi, and Kalla [10] developed its fundamental properties and applications systematically.

Elzaki transform. Elzaki [9] proposed $E\{f\}(v) = v \int_0^\infty e^{-t/v} f(t) dt$ and applied it to ordinary differential equations with variable coefficients that resist the Sumudu form.

Aboodh transform. Aboodh [11] introduced $A\{f\}(v) = \frac{1}{v} \int_0^\infty e^{-vt} f(t) dt$, designed so that the images of derivatives take a particularly simple divided form.

Mahgoub and Mohand transforms. Mahgoub [12] proposed the Laplace-Carson-type transform $M\{f\}(v) = v \int_0^\infty e^{-vt} f(t) dt$; Mohand and Mahgoub [13] proposed $\mathcal{M}\{f\}(v) = v^2 \int_0^\infty e^{-vt} f(t) dt$.

Aggarwal and co-workers subsequently produced an extensive series of application papers using these kernels for Volterra integral equations, integro-differential equations, and mixture problems [20–23].

Kamal and Sawi transforms. Kamal (Sedeeg) [14] proposed $K\{f\}(v) = \int_0^\infty e^{-t/v} f(t) dt$; the Sawi transform [15] uses $\frac{1}{v^2} \int_0^\infty e^{-t/v} f(t) dt$.

Shehu and Natural transforms. The Natural transform of Khan and Khan [18] and the Shehu transform of Maitama and Zhao [16] are two-variable transforms, $\int_0^\infty e^{-st/u} f(t) dt$ up to prefactors, that reduce to the Laplace and Sumudu transforms on the diagonal $u = 1$ and $s = 1$ respectively. They have been applied extensively to fractional differential equations.

Sadik transform. Shaikh [17] introduced a two-exponent transform $\frac{1}{v^\beta} \int_0^\infty e^{-v^\alpha t} f(t) dt$ and observed that many one-parameter transforms are special cases. The Sadik transform is, to our knowledge, the closest published precursor of the framework developed here; the G.G. integral transform strictly contains it (Section 10).

Further variants. The literature also contains the Tarig, Pourreza, SEE, Rishi, Soham, Kashuri-Fundo, Formable [24], Kharrat-Toma, Anju, Iman, Emad-Falih, Sara, Rangaig, Gupta (and Rohit and Dinesh) [25], Z.Z., NE, Ramadan Group, and complex-parameter (Sherifa-Emad-Ali, complex Sadik) transforms, among others. Kim and co-workers [26] studied “Laplace-typed” transforms $v^\alpha \int_0^\infty e^{-t/v} f(t) dt$ as a one-parameter family, and Jafari [19] introduced a general transform $\varphi(v) \int_0^\infty e^{-(v)t} f(t) dt$ with arbitrary positive functions φ , from which particular transforms follow by choosing the two functions.

2.3 Applications literature

Across this family, the demonstrated applications are remarkably uniform: first and second order linear ordinary differential equations from RL, RC, and RLC circuits [21, 27], Newtonian cooling and heat problems [22], exponential population growth and radioactive decay [23, 28], damped and forced mechanical oscillators [1, 5], Volterra integral and integral-differential equations [20], and, increasingly, fractional-order differential equations under Caputo and Riemann-Liouville derivatives [16, 29–31]. This uniformity of application is itself evidence that the transforms are operationally equivalent, and it motivates a framework in which that equivalence is a theorem rather than an empirical observation.

2.4 Synthesis

Three observations emerge from the review. First, every transform in Section 2.2 has a kernel of the exact form $p v^m e^{-qv^w t}$ for some constants (p, q, m, w) , possibly after renaming the transform variable. Second, the operational properties published for each transform are structurally identical, differing only in bookkeeping factors of v . Third, no published work provides, for the full power-law family, a rigorous and complete treatment of existence, convergence region, inversion, and uniqueness together with quantified numerical validation. Jafari's general transform [19] is more general in allowing arbitrary φ , but for that reason it does not exhibit the explicit four-parameter dictionary, and published treatments of it do not include the error analysis and complexity comparison presented here.

3. Motivation and Research Gap

The gap addressed by this study is therefore threefold.

(G1) Unification gap. No existing publication presents a single explicitly parameterized transform, with a complete parameter dictionary, that recovers the full catalogue of power-law Laplace-type transforms (thirty-eight are catalogued here) as verifiable special cases.

(G2) Rigor gap. The papers introducing individual transforms typically state operational properties with computational verifications but without formal existence, convergence, inversion, and uniqueness theorems under explicit hypotheses. A framework intended to subsume them must supply this rigor once and for all.

(G3) Validation gap. The applications literature presents worked analytical examples but rarely quantifies accuracy against independent numerical solutions or discusses computational cost. A modern presentation requires validated error tables and a complexity discussion.

4. Research Contributions

The contributions of this study are:

1. Definition of the four-parameter G.G. integral transform (1.1) with an explicitly characterized admissible parameter set (Section 6).
2. A unification dictionary (Tables 1 and 2) recovering thirty-eight published transforms by parameter specialization, together with an interpreted comparative study (Table 3).
3. An existence and convergence theorem with an explicit convergence region $qv^w > \sigma_0$ and a growth bound on the image (Theorem 7.1), plus the asymptotic annihilation property (Theorem 7.2).
4. Complete proofs of the operational calculus: linearity, elementary transforms, first and second shifting, change of scale, derivatives of all orders, and multiplication by t^n (Theorems 8.1 to 8.7).
5. A rigorous inversion theorem via the Bromwich contour under the substitution $s = qv^w$ (Theorem 9.1), a uniqueness theorem of Lerch type (Theorem 9.2), and a convolution theorem (Theorem 9.3) that exposes the exact prefactor $1/(pv^m)$ obstruction and its resolution.
6. Worked engineering applications with physical discussion: LR and RLC circuits, Newtonian cooling, melting and conversion kinetics, bacterial and demographic growth, radioactive half-life, free, damped, critically damped, and overdamped vibration, and terminal velocity of a falling body (Section 11).
7. Numerical validation of every closed-form solution against adaptive Runge-Kutta integration with tabulated absolute error, relative error, RMSE, and maximum error (Sections 12 and 13).
8. A computational complexity analysis, including a symbolic-computation benchmark, comparing the transform-domain route with direct symbolic solution (Section 14).
9. Publication-quality redrawn figures for all circuits, mechanical systems, and response curves.

5. Mathematical Preliminaries

Definition 5.1 (Piecewise continuity). A function $f: [0, \infty) \rightarrow \mathbb{R}$ is piecewise continuous on $[0, \infty)$ if on every finite interval $[0, T]$ it has at most finitely many discontinuities, each of jump type, and is continuous elsewhere.

Definition 5.2 (Exponential order). A function f is of exponential order $\sigma_0 \in \mathbb{R}$ if there exist constants $M > 0$ and $T \geq 0$ such that

$$|f(t)| \leq M e^{\sigma_0 t} \quad \text{for all } t \geq T. \quad (5.1)$$

We write $f \in \mathcal{E}(\sigma_0)$ and denote by $\mathcal{A}$ the class of functions that are piecewise continuous on $[0, \infty)$ and of exponential order for some σ_0 .

Definition 5.3 (Gamma function). $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ for $\text{Re } z > 0$, with $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$.

Definition 5.4 (Convolution). For $f, g \in \mathcal{A}$, $(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau$.

Lemma 5.5 (Laplace correspondence). Let $F(s) = \mathcal{L}\{f\}(s)$ denote the Laplace transform of $f \in \mathcal{A}$, absolutely convergent for $\text{Re } s > \sigma_0$. Then formally, whenever $s = qv^w > \sigma_0$,

$$\{f\}(v) = p v^m F(qv^w). \quad (5.2)$$

Proof. Immediate from Definition (1.1) by reading qv^w as the Laplace variable.

Identity (5.2) is the structural backbone of the paper: it converts every question about the G.G. integral transform into a question about the Laplace transform evaluated along the curve $s = qv^w$, dressed by the invertible prefactor $p v^m$. All proofs below are nevertheless carried out directly from Definition (1.1), so that the theory is self-contained, and (5.2) is used as an independent check.

6. Definition of the G.G. Integral Transform

Definition 6.1 (G.G. integral transform). Let $f \in \mathcal{A}$ with exponential order σ_0 , and let the parameters satisfy

$$p > 0, \quad q > 0, \quad m \in \mathbb{R}, \quad w \in \mathbb{R} \setminus \{0\}. \quad (6.1)$$

The G.G. integral transform of f is

$$\{f(t)\}(v) = G(v) = p v^m \int_0^\infty e^{-qv^w t} f(t) dt, \quad (6.2)$$

defined for all $v > 0$ such that $qv^w > \sigma_0$.

Remark 6.2 (Admissible domain). For $w > 0$ the condition $qv^w > \sigma_0$ defines an interval $v > (\sigma_0/q)^{1/w}$ when $\sigma_0 > 0$, and all $v > 0$ when $\sigma_0 \leq 0$. For $w < 0$ it defines $0 < v < (\sigma_0/q)^{1/w}$ when $\sigma_0 > 0$, and all $v > 0$ when $\sigma_0 \leq 0$. In every case the domain is a nonempty open subinterval of $(0, \infty)$, which suffices for the uniqueness theory of Section 9.

Remark 6.3 (Two-variable specializations). Transforms with two variables, such as the Natural, Shehu, Z.Z., NE, and Formable transforms, are recovered by allowing p and q to depend on the second variable; for example, the Shehu transform corresponds to $p = 1$, $q = s/u$ treated as the effective exponential rate. Table 2 records these identifications.

Remark 6.4 (Dimensional bookkeeping). If t carries units of time, then qv^w carries units of inverse time, and the prefactor $p v^m$ controls the units of the image. The Sumudu specialization $(1, 1, -1, -1)$ is precisely the choice that makes the image carry the same units as f , which explains its popularity in control applications [7, 10]; the G.G. framework exposes units-preservation as a one-line parameter condition, $m = w$ with $q = 1$, rather than a special feature of one transform.

7. Existence and Convergence Analysis

Theorem 7.1 (Existence, convergence region, and growth bound). Let $f \in \mathcal{A}$ be piecewise continuous on $[0, \infty)$ and of exponential order σ_0 , so that $|f(t)| \leq M e^{\sigma_0 t}$ for $t \geq T$. Then for every $v > 0$ satisfying

$$qv^w > \sigma_0, \quad (7.1)$$

the integral in (6.2) converges absolutely, and the image obeys the growth bound

$$|G(v)| \leq p v^m \left(C_T + \frac{M e^{-(qv^w - \sigma_0)T}}{qv^w - \sigma_0} \right), \quad C_T = \int_0^T |f(t)| dt < \infty. \quad (7.2)$$

Proof. Split the integral at T . On $[0, T]$, f is piecewise continuous, hence bounded and integrable, and $e^{-qv^w t} \leq \max(1, e^{-qv^w T})$ is bounded, so the first piece converges and is bounded by $C_T \max(1, e^{-qv^w T}) \leq C_T$ for $qv^w > 0$; when $\sigma_0 \geq 0$ condition (7.1) already forces $qv^w > 0$, and when $\sigma_0 < 0$ and $qv^w \leq 0$ the bound is adjusted by the constant factor $e^{-qv^w T}$ without affecting convergence. On $[T, \infty)$,

$$\int_T^\infty |e^{-qv^w t} f(t)| dt \leq M \int_T^\infty e^{-(qv^w - \sigma_0)t} dt = \frac{M e^{-(qv^w - \sigma_0)T}}{qv^w - \sigma_0},$$

which is finite precisely when (7.1) holds. Absolute convergence of both pieces gives absolute convergence of (6.2), and multiplying by pv^m gives (7.2).

Corollary 7.1.1. Every bounded piecewise continuous function ($\sigma_0 = 0$), every polynomial ($\sigma_0 = \varepsilon$ for every $\varepsilon > 0$), and every function $e^{kt}h(t)$ with h bounded ($\sigma_0 = k$) possesses a G.G. integral transform on the region (7.1). The Gaussian-type function e^{t^2} is of no finite exponential order and has no G.G. integral transform, exactly as it has no Laplace transform.

Theorem 7.2 (Asymptotic annihilation). Under the hypotheses of Theorem 7.1,

$$\lim_{qv^w \rightarrow \infty} \frac{G(v)}{pv^m} = 0. \quad (7.3)$$

Consequently, a candidate image for which $G(v)/(pv^m)$ does not vanish as $qv^w \rightarrow \infty$ (for instance a nonzero constant multiple of pv^m , or pv^m times a polynomial in qv^w) cannot be the G.G. integral transform of any function in $\mathcal{A}$.

Proof. From the proof of Theorem 7.1, $|G(v)/(pv^m)| \leq C_T e^{\max(0, -qv^w)T} + M e^{-(qv^w - \sigma_0)T} / (qv^w - \sigma_0)$. As $qv^w \rightarrow \infty$ the second term vanishes; for the first term, refine the split point: for any $\varepsilon > 0$ choose δ so small that $\int_0^\delta |f| dt < \varepsilon$, and note $\int_\delta^T e^{-qv^w t} |f(t)| dt \leq e^{-qv^w \delta} \int_\delta^T |f| dt \rightarrow 0$. Hence

the limit superior of the modulus is at most ε for every $\varepsilon > 0$.

Remark 7.3 (Analyticity). By identity (5.2) and the analyticity of the Laplace transform in its half-plane of convergence, $G(v)/(pv^m)$ is real-analytic in v on the open region (7.1), and extends to an analytic function of the complex variable $s = qv^w$ on $\text{Res} > \sigma_0$. This is the property that powers the uniqueness and inversion theory of Section 9.

8. Fundamental Properties

Throughout this section $f, f_1, f_2 \in \mathcal{A}$, all stated identities hold on the common region of convergence, and we abbreviate $s \equiv qv^w$ where convenient.

8.1 Transforms of elementary functions

Theorem 8.1. For v in the region (7.1) applicable to each entry:

#	$f(t)$	$\{f(t)\}(v)$	Validity
1	a (constant)	$\frac{a p v^m}{qv^w}$	$qv^w > 0$
2	$t^n, n \in \mathbb{N}_0$	$\frac{n! p v^m}{(qv^w)^{n+1}}$	$qv^w > 0$
3	$t^v, v > -1$	$\frac{\Gamma(v+1) p v^m}{(qv^w)^{v+1}}$	$qv^w > 0$
4	e^{kt}	$\frac{p v^m}{qv^w - k}$	$qv^w > k$
5	$\text{Sink}t$	$\frac{k p v^m}{q^2 v^{2w} + k^2}$	$qv^w > 0$
6	$\cos kt$	$\frac{p q v^{m+w}}{q^2 v^{2w} + k^2}$	$qv^w > 0$
7	$\sinh kt$	$\frac{k p v^m}{q^2 v^{2w} - k^2}$	$qv^w > k $
8	$\cosh kt$	$\frac{p q v^{m+w}}{q^2 v^{2w} - k^2}$	$qv^w > k $

Proof. (1) With the substitution $x = qv^w t$, $dt = dx/(qv^w)$,

$$\{a\} = ap v^m \int_0^\infty e^{-qv^w t} dt = \frac{ap v^m}{qv^w} \int_0^\infty e^{-x} dx = \frac{ap v^m}{qv^w}.$$

(2), (3) The same substitution gives

$$\{tv\} = p v^m \int_0^\infty t^v e^{-qv^w t} dt = \frac{p v^m}{(qv^w)^{v+1}} \int_0^\infty x^v e^{-x} dx = \frac{\Gamma(v+1) p v^m}{(qv^w)^{v+1}},$$

and $\Gamma(n+1) \stackrel{0}{=} n!$ yields (2). (4) $\{e^{kt}\} = p v^m \int_0^\infty e^{-(qv^w-k)t} dt = \frac{p v^m}{qv^w-k}$ for $qv^w > k$. (5), (6)

Using $\sin kt = \frac{1}{2i}(e^{ikt} - e^{-ikt})$, linearity (Theorem 8.2, whose proof is independent), and entry (4) extended to complex rates with $\text{Re}(qv^w) > 0$,

$$\{\sin kt\} = \frac{p v^m}{2i} \left(\frac{1}{qv^w - ik} - \frac{1}{qv^w + ik} \right) = \frac{p v^m}{2i} \cdot \frac{2ik}{q^2 v^{2w} + k^2} = \frac{k p v^m}{q^2 v^{2w} + k^2}$$

$$\{\cos kt\} = \frac{p v^m}{2} \left(\frac{1}{qv^w - ik} + \frac{1}{qv^w + ik} \right) = \frac{p v^m}{q^2 v^{2w} + k^2} = \frac{p q v^{m+w}}{q^2 v^{2w} + k^2}.$$

(7), (8) Identical computations with $\sinh kt = \frac{1}{2}(e^{kt} - e^{-kt})$ and $\cosh kt = \frac{1}{2}(e^{kt} + e^{-kt})$, valid for $qv^w > |k|$.

8.2 Linearity

Theorem 8.2 (Linearity). For constants a, b and $f_1, f_2 \in \mathcal{A}$,

$$\{af_1(t) + bf_2(t)\} = a \{f_1(t)\} + b \{f_2(t)\} \quad (8.1)$$

on the intersection of the two regions of convergence.

Proof. Both integrals converge absolutely on the common region (Theorem 7.1), so the integral of the linear combination splits:

$$p v^m \int_0^\infty e^{-qv^w t} [af_1 + bf_2] dt = a p v^m \int_0^\infty e^{-qv^w t} f_1 dt + b p v^m \int_0^\infty e^{-qv^w t} f_2 dt.$$

8.3 Shifting and scaling

Theorem 8.3 (First shifting theorem). If $\{f(t)\}(v) = G(v) = p v^m \Phi(qv^w)$ where $\Phi(s) = \int_0^\infty e^{-st} f(t) dt$, then

$$\{e^{kt} f(t)\}(v) = p v^m \Phi(qv^w - k), \quad qv^w > \sigma_0 + k. \quad (8.2)$$

In words: multiplication by e^{kt} in the time domain replaces qv^w by $qv^w - k$ in the image, leaving the prefactor $p v^m$ untouched.

Proof.

$$\{e^{kt} f(t)\} = p v^m \int_0^\infty e^{-qv^w t} e^{kt} f(t) dt = p v^m \int_0^\infty e^{-(qv^w-k)t} f(t) dt = p v^m \Phi(qv^w - k).$$

Corollary 8.3.1. With $r \equiv qv^w - k$:

$$\{e^{kt} t^n\} = \frac{n! p v^m}{r^{n+1}}, \quad \{e^{kt} \sin bt\} = \frac{b p v^m}{r^2 + b^2}, \quad \{e^{kt} \cos bt\} = \frac{p v^m r}{r^2 + b^2},$$

$$\{e^{kt} \sinh bt\} = \frac{b p v^m}{r^2 - b^2}, \quad \{e^{kt} \cosh bt\} = \frac{p v^m r}{r^2 - b^2}.$$

Proof. Apply (8.2) to each entry of Theorem 8.1.

Theorem 8.4 (Second shifting theorem). Let $a > 0$ and define the delayed function

$$f_a(t) = \begin{cases} 0, & 0 \leq t < a, \\ f(t-a), & t \geq a. \end{cases}$$

Then

$$\{f_a(t)\}(v) = e^{-a q v^w} \{f(t)\}(v). \quad (8.3)$$

Proof. Substituting $u = t - a$,

$$\{f_a\} = p v^m \int_a^\infty e^{-qv^w t} f(t-a) dt = p v^m \int_0^\infty e^{-qv^w(u+a)} f(u) du = e^{-a q v^w} \{f\}.$$

Theorem 8.5 (Change of scale). For $a > 0$, writing $G(v) = p v^m \Phi(qv^w)$ as in Theorem 8.3,

$$\{f(at)\}(v) = \frac{p v^m}{a} \Phi\left(\frac{qv^w}{a}\right). \quad (8.4)$$

Proof. With $u = at$,

$$\{f(at)\} = pv^m \int_0^\infty e^{-qv^w t} f(at) dt = \frac{pv^m}{a} \int_0^\infty e^{-\frac{qv^w}{a} u} f(u) du = \frac{pv^m}{a} \Phi\left(\frac{qv^w}{a}\right).$$

8.4 Transforms of derivatives

Theorem 8.6 (Derivatives). Let $f, f', \dots, f^{(n-1)}$ be continuous on $[0, \infty)$ and $f^{(n)}$ piecewise continuous, with all of them in $\mathcal{E}(\sigma_0)$. Then for $qv^w > \max(\sigma_0, 0)$:

$$\{f'(t)\} = qv^w G(v) - pv^m f(0), \quad (8.5)$$

$$\{f''(t)\} = q^2 v^{2w} G(v) - pqv^{m+w} f(0) - pv^m f'(0), \quad (8.6)$$

and in general

$$\{f^{(n)}(t)\} = (qv^w)^n G(v) - pv^m \sum_{j=0}^{n-1} (qv^w)^{n-1-j} f^{(j)}(0). \quad (8.7)$$

Proof. For (8.5), integrate by parts:

$$\{f'\} = pv^m \int_0^\infty e^{-qv^w t} f'(t) dt = pv^m [e^{-qv^w t} f(t)]_0^\infty + pv^m qv^w \int_0^\infty e^{-qv^w t} f(t) dt.$$

Since $f \in \mathcal{E}(\sigma_0)$ and $qv^w > \sigma_0$, $e^{-qv^w t} f(t) \rightarrow 0$ as $t \rightarrow \infty$, so the bracket equals $-f(0)$ and (8.5) follows. For (8.6), apply (8.5) to f' in place of f :

$$\{f''\} = qv^w \{f'\} - pv^m f'(0) = qv^w [qv^w G(v) - pv^m f(0)] - pv^m f'(0),$$

which is (8.6). The general formula (8.7) follows by induction: assuming (8.7) for n , apply (8.5) to $f^{(n)}$ to obtain

$$\{f^{(n+1)}\} = qv^w \{f^{(n)}\} - pv^m f^{(n)}(0) = (qv^w)^{n+1} G(v) - pv^m \sum_{j=0}^n (qv^w)^{n-j} f^{(j)}(0).$$

Remark 8.6.1. Formulas (8.5) to (8.7) reduce, under the parameter dictionary of Table 2, to the published derivative rules of every transform in the family; for example $(p, q, m, w) = (1, 1, 0, 1)$ gives the Laplace rule $sF(s) - f(0)$ and $(1, 1, -1, -1)$ gives the Sumudu rule $[S(v) - f(0)]/v$.

8.5 Multiplication by powers of t

Theorem 8.7 (Multiplication by t^n). Write $G(v) = pv^m \Phi(s)$ with $s = qv^w$ as before, where $\Phi(s) = \int_0^\infty e^{-st} f(t) dt$. Then for each $n \in \mathbb{N}$,

$$\{t^n f(t)\}(v) = (-1)^n p v^m \left[\frac{d^n \Phi}{ds^n} \right]_{s=qv^w}. \quad (8.8)$$

Equivalently, in terms of the raw variable v (for $n = 1$), by the chain rule $\frac{d\Phi}{dv} = \frac{d\Phi}{ds} qwv^{w-1}$,

$$\{tf(t)\}(v) = \frac{m}{qwv^w} G(v) - \frac{v^{1-w}}{qw} \frac{dG}{dv}. \quad (8.9)$$

Proof. Differentiation under the integral sign, justified by the locally uniform absolute convergence established in Theorem 7.1 and the dominated convergence theorem, gives

$$\frac{d^n \Phi}{ds^n} = \int_0^\infty \frac{\partial^n}{\partial s^n} e^{-st} f(t) dt = (-1)^n \int_0^\infty t^n e^{-st} f(t) dt,$$

and multiplying by $(-1)^n p v^m$ yields (8.8). For (8.9), differentiate $G(v) = pv^m \Phi(qv^w)$ with respect to v :

$$\frac{dG}{dv} = m p v^{m-1} \Phi + p v^m \Phi'(s) q w v^{w-1} = \frac{m}{v} G(v) - q w v^{w-1} \{tf(t)\} \cdot \frac{1}{1}$$

using (8.8) with $n = 1$ in the last step; solving for $\{tf(t)\}$ gives (8.9).

Corollary 8.7.1 (Image of $tf'(t)$). Combining (8.9) with (8.5),

$$\{tf'(t)\}(v) = \frac{m}{qwv^w} [qv^w G(v) - pv^m f(0)] - \frac{v^{1-w}}{qw} \frac{d}{dv} [qv^w G(v) - pv^m f(0)],$$

which is the identity needed for differential equations with coefficients linear in t (for example the Laguerre and Bessel families).

9. New Theoretical Results: Inversion, Uniqueness, and Convolution

9.1 The inverse transform

Theorem 9.1 (Inversion theorem). Let $f \in \mathcal{A}$ be continuous on $[0, \infty)$ with exponential order σ_0 , and let $G(v) = \{f\}(v)$. Define

$$\Phi(s) = \frac{G(v(s))}{p(v(s))}, \quad v(s) = \left(\frac{s}{q}\right)^{1/w}, \quad (9.1)$$

which by Remark 7.3 extends analytically to the half-plane $\text{Re } s > \sigma_0$. Then for every $c > \sigma_0$ and every $t > 0$,

$$f(t) = -1\{G(v)\}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \Phi(s) ds, \quad (9.2)$$

the integral being understood as a Cauchy principal value; at a jump discontinuity the right side converges to $1/2 [f(t^+) + f(t^-)]$.

Proof. By construction $\Phi(s) = \int_0^\infty e^{-st} f(t) dt$ is precisely the Laplace transform of f evaluated on $\text{Re } s > \sigma_0$: indeed, from Definition 6.1, $G(v) = pv^m \int_0^\infty e^{-qv^wt} f(t) dt$, and dividing by pv^m and substituting $s = qv^w$, which is a bijection of the admissible v -interval onto (σ_0, ∞) because $w \neq 0$ makes $v \mapsto qv^w$ strictly monotone, gives $\Phi(s) = \int_0^\infty e^{-st} f(t) dt$ on a real interval, and hence on the half-plane by analytic continuation. The representation (9.2) is then the classical Bromwich-Mellin inversion integral for the Laplace transform, valid under the stated hypotheses [2, 3].

Remark 9.1.1 (Practical inversion). In applications one rarely evaluates (9.2) directly. Instead, one writes the image in the form $pv^m \times (\text{rational function of } qv^w)$, decomposes the rational function into partial fractions in the variable $s = qv^w$, and reads each term against Theorem 8.1 and Corollary 8.3.1. Every inversion in Section 11 is carried out this way, and Theorem 9.2 guarantees the answer is the only continuous one.

Example 9.1.2. Given $G(v) = \frac{10pqv^{m+w} + 40pv^m}{q^2v^{2w} + 4qv^w + 4}$, factor the denominator as $(qv^w + 2)^2$ and decompose:

$$\frac{G(v)}{pv^m} = \left[\frac{10s + 40}{(s + 2)^2} \right]_{s=qv^w} = \left[\frac{10}{s + 2} + \frac{20}{(s + 2)^2} \right]_{s=qv^w},$$

whence $f(t) = 10e^{-2t} + 20te^{-2t} = 10e^{-2t}(1 + 2t)$ by Theorem 8.1(4) and Corollary 8.3.1.

9.2 Uniqueness

Theorem 9.2 (Uniqueness, Lerch-type). Let $f_1, f_2 \in \mathcal{A}$ be continuous on $[0, \infty)$ and suppose $\{f_1\}(v) = \{f_2\}(v)$ for all v in some open subinterval of the common admissible domain. Then $f_1(t) = f_2(t)$ for every $t \geq 0$. If f_1, f_2 are merely piecewise continuous, they agree at every common point of continuity.

Proof. Let $h = f_1 - f_2 \in \mathcal{A}$ and let $\Phi_h(s)$ denote its Laplace transform. Equality of the G.G. images on an open v -interval, after division by the nonvanishing factor pv^m and the monotone substitution $s = qv^w$, gives $\Phi_h(s) = 0$ on an open real s -interval inside $\text{Re } s > \sigma_0$. Since Φ_h is analytic on that half-plane (Remark 7.3), the identity theorem forces $\Phi_h \equiv 0$ there. Lerch's theorem [2, 3] then yields $h(t) = 0$ at every point of continuity, hence everywhere when h is continuous.

Remark 9.2.1. Theorem 9.2 is what licenses the phrase “taking the inverse G.G. integral transform” in Section 11: the continuous preimage recovered by partial fractions is the unique continuous preimage.

9.3 Convolution

Theorem 9.3 (Convolution theorem). For $f, g \in \mathcal{A}$,

$$\{(f * g)(t)\}(v) = \frac{1}{pv^m} \{f\}(v) \{g\}(v). \quad (9.3)$$

Proof. By absolute convergence (Theorem 7.1) Fubini's theorem applies:

$$\{f * g\} = pv^m \int_0^\infty e^{-qv^wt} \int_0^t f(\tau) g(t - \tau) d\tau dt = pv^m \int_0^\infty f(\tau) \int_\tau^\infty e^{-qv^wt} g(t - \tau) dt d\tau,$$

and, substituting $u = t - \tau$ in the inner integral,

$$\{f * g\} = p v^m \int_0^\infty f(\tau) e^{-q v^w \tau} d\tau \int_0^\infty e^{-q v^w u} g(u) du = \frac{1}{p v^m} [p v^m \Phi_f][p v^m \Phi_g],$$

which is (9.3).

Remark 9.3.1 (The prefactor obstruction). Unlike the Laplace case, the G.G. image of a convolution is not the plain product of the images: the correction $1/(p v^m)$ is forced by the prefactor and disappears exactly when $p v^m = 1$, that is, for the Laplace-type members with $p = 1$, $m = 0$ (Laplace, Kamal, Shehu with u absorbed). Published convolution rules for the Elzaki transform, image product divided by v , and the Sumudu transform, image product times v , are the specializations $(p, m) = (1, 1)$ and $(1, -1)$ of (9.3). The general statement makes the pattern visible at a glance.

Corollary 9.3.2 (Integral rule). Taking $g \equiv 1$ in (9.3) and using Theorem 8.1(1),

$$\int_0^t f(\tau) d\tau \cdot (v) = \frac{\{f\}(v)}{q v^w},$$

the transform-domain antiderivative rule, used for integro-differential equations.

10. Relationship with Existing Transforms: The Unification Dictionary

10.1 Kernel catalogue

Table 1 catalogues the kernels of the principal published Laplace-type transforms. Throughout, v denotes the transform variable of the cited work (renamed where necessary), and s, u denote the second variable of two-variable transforms.

Table 1. Existing Laplace-type integral transforms and their kernels.

No.	Transform	Kernel $k(t, v)$	Introduced in / representative source
1	Laplace	e^{-vt}	[1–3]
2	Mahgoub (Laplace-Carson)	$v e^{-vt}$	[12]
3	Mohand	$v^2 e^{-vt}$	[13]
4	Rohit	$v^3 e^{-vt}$	[25]
5	Dinesh	$v^5 e^{-vt}$	[25]
6	Aboodh	$v^{-1} e^{-vt}$	[11]
7	Gupta	$v^{-3} e^{-vt}$	[25]
8	Kamal	$e^{-t/v}$	[14]
9	Elzaki	$v e^{-t/v}$	[9]
10	Sadik	$v^{-\beta} e^{-v^\alpha t}$	[17]
11	Sumudu	$v^{-1} e^{-t/v}$	[7, 10]
12	Sawi	$v^{-2} e^{-t/v}$	[15]
13	Tarig	$v^{-1} e^{-t/v^2}$	[32]
14	Shehu	$e^{-st/u}$	[16]
15	Natural	$u^{-1} e^{-st/u}$	[18]
16	Z.Z.	$s u^{-1} e^{-st/u}$	[33]
17	Laplace-typed (G_α)	$v^\alpha e^{-t/v}$	[26]
18	Pourreza	$v^2 e^{-v^2 t}$	[34]
19	α -integral Laplace	e^{-t/v^α}	[26]
20	SEE	$v^{-n} e^{-vt}$	[35]
21	NE	$(su)^{-1} e^{-st/u}$	[36]
22	Formable	$s u^{-1} e^{-st/u}$	[24]
23	Soham	$v^{-1} e^{-v^\alpha t}$	[37]
24	Kashuri-Fundo	$v^{-1} e^{-t/v^2}$	[38]
25	Ramadan Group	$v e^{-st}$ (two-variable)	[39]
26	Emad-Sara	$v^{-2} e^{-vt}$	[40]
27	Complex SEE	$v^{-n} e^{-vt}$, v complex	[35]

28	Sum	$v^{-r}e^{-vt}$	[41]
29	Iman	$v^{-2}e^{-v^2t}$	[42]
30	Anju	$v^2e^{-t/v}$	[43]
31	Emad-Falih	$v e^{-t/v^2}$	[40]
32	GF1	$v^3e^{-t/v}$	[44]
33	GF2	$v^{-5}e^{-vt}$	[44]
34	Sherifa-Emad-Ali (SEA)	$s v^{-1}e^{-isvt}$	[45]
35	Kushare	$v e^{-v^\alpha t}$	[46]
36	Complex Sadik	$i v^{-1}e^{-iv^\alpha t}$	[17]
37	Rishi	$v s^{-1}e^{-st/v}$	[47]
38	Kharrat-Toma	v^3e^{-t/v^2}	[48]

Note. Bibliographic details for several of the more recently proposed transforms (entries 20 to 38) appear in regional and emerging journals with variable indexing; the kernels listed follow the forms reported in the secondary literature and in the compilation of the present authors. The unification claims below depend only on the kernel forms, not on the bibliographic provenance.

10.2 Parameter dictionary

Every kernel in Table 1 has the form $p v^{me-qv^wt}$, possibly with p or q carrying the second variable of a two-variable transform, or a complex constant. Table 2 records the identification.

Table 2. Recovery of existing transforms from the G.G. integral transform
 $\{f\} = p v^m \int_0^\infty e^{-qv^wt} f(t) dt.$

No.	p	q	m	w	Recovered transform
1	1	1	0	1	Laplace
2	1	1	1	1	Mahgoub (Laplace-Carson)
3	1	1	2	1	Mohand
4	1	1	3	1	Rohit
5	1	1	5	1	Dinesh
6	1	1	-1	1	Aboodh
7	1	1	-3	1	Gupta
8	1	1	0	-1	Kamal
9	1	1	1	-1	Elzaki
10	1	1	$-\beta$	α	Sadik
11	1	1	-1	-1	Sumudu
12	1	1	-2	-1	Sawi
13	1	1	-1	-2	Tarig
14	1	s/u	0	$1, v = u$	Shehu
15	$1/u$	s/u	0	$1, v = u$	Natural
16	s/u	s/u	0	$1, v = u$	Z.Z.
17	1	1	α	-1	Laplace-typed G_α
18	1	1	2	2	Pourreza
19	1	1	0	$-\alpha$	α -integral Laplace
20	1	1	$-n$	1	SEE
21	$1/(su)$	s/u	0	$1, v = u$	NE
22	s/u	s/u	0	$1, v = u$	Formable
23	1	1	-1	α	Soham
24	1	1	-1	-2	Kashuri-Fundo
25	1	s	1	0^+ (fixed rate)	Ramadan Group
26	1	1	-2	1	Emad-Sara
27	1	1	$-n$	$1, v \in \mathbb{C}$	Complex SEE
28	1	1	$-r$	1	Sum

29	1	1	-2	2	Iman
30	1	1	2	-1	Anju
31	1	1	1	-2	Emad-Falih
32	1	1	3	-1	GF1
33	1	1	-5	1	GF2
34	s	is	-1	1	SEA
35	1	1	1	α	Kushare
36	i	i	-1	α	Complex Sadik
37	v/s	s	0	-1	Rishi
38	1	1	3	-2	Kharrat-Toma

Proposition 10.1. For every row of Table 2 with real parameters satisfying (6.1), all theorems of Sections 7 to 9 hold verbatim for the recovered transform, with $s = qv^w$ the effective Laplace variable. For rows with complex or two-variable parameters, the theorems hold after the corresponding analytic continuation, with the second variable treated as a fixed positive parameter.

Proof. Immediate, since every proof in Sections 7 to 9 used only $p \neq 0$, qv^w real and exceeding σ_0 on an open interval, and the monotonicity of $v \rightarrow qv^w$; two-variable cases fix s (or u) and vary the remaining variable.

10.3 Interpreted comparative study

Table 2 is not merely a list; it explains the structure of the family. Three interpretive conclusions follow.

(i) The m -axis is cosmetic; the w -axis is substantive. Transforms differing only in m (Laplace, Mahgoub, Mohand, Rohit, Dinesh, Aboodh, Gupta, SEE, Sum, GF2 all have $w = 1$) are related by multiplication by a fixed power of v , an invertible operation that alters no analytic content: convergence regions, inversion contours, and uniqueness classes are identical. Differences in w (Laplace $w = 1$ versus Kamal-Elzaki-Sumudu $w = -1$ versus Tarig-Kashuri $w = -2$ versus Pourreza $w = 2$) reparameterize the Laplace half-plane by the substitution $s = qv^w$; this changes which limits ($v \rightarrow 0$ or $v \rightarrow \infty$) probe the large- s regime, and hence which Tauberian statements look natural, but again preserves all analytic content.

(ii) Exactly one member is units-preserving. Requiring the image of f to carry the units of f forces $m = w$ and $q = 1$ with p dimensionless; the choice $(1, 1, -1, -1)$, the Sumudu transform, is the canonical solution, which recovers, as a parameter computation, the property Watugala advertised [7].

(iii) Comparative summary. Table 3 compares the main representatives across the criteria that matter in applications.

Table 3. Comparative study of representative transforms as specializations of the G.G. integral transform.

Criterion	Laplace	Sumudu	Elzaki	Aboodh	Mohand	Shehu	Sadik	G.G. (this work)
Parameters (p, q, m, w)	$(1, 1, 0, 1)$	$(1, 1, -1, -1)$	$(1, 1, 1, -1)$	$(1, 1, -1, 1)$	$(1, 1, 2, 1)$	$(1, \frac{s}{u}, 0, 1)$	$(1, 1, -\beta, \alpha)$	free
Convergence region	$s > \sigma_0$	$0 < v < 1/\sigma_0$	$0 < v < 1/\sigma_0$	$v > \sigma_0$	$v > \sigma_0$	$s/u > \sigma_0$	$v^\alpha > \sigma_0$	$qv^w > \sigma_0$
Derivative rule	$sF - f(0)$	$\frac{S - f(0)}{v}$	$\frac{E}{v} - vf(0)$	$vA - \frac{f(0)}{v}$	$vM - v^2f(0)$	$\frac{s}{u}\Sigma - f(0)$	$v^\alpha\Psi - \frac{f(0)}{v^\beta}$	Eq. (8.5)
Convolution prefactor	1	v	$1/v$	v	$1/v^2$	1	v^β	$1/(pv^m)$, Eq. (9.3)

Units-preserving	no	yes	No	no	no	no	no	iff $m = w, q = 1$
Unifies others	no	no	No	no	no	Laplace, Sumudu	one-exponent family	all 38 of Table 2
Typical applications	ODE/PDE, control	control, PDE	variable-coefficient ODE	integral eqs.	Volterra eqs.	fractional ODE	family analysis	all of the foregoing

The conclusion of the comparison is the central claim of the paper: under appropriate parameter selection the G.G. integral transform reproduces each competitor exactly, so its operational calculus is a common generalization, and any efficiency or convenience claimed for a particular member is available inside the G.G. framework by choosing its row of Table 2.

11. Engineering Applications

This section solves canonical initial value problems from electrical engineering, thermal science, chemical and biological kinetics, and mechanics using the G.G. integral transform. Every solution follows the same three-step pattern: (i) transform the governing equation using Theorem 8.6 and linearity; (ii) solve the resulting algebraic equation for the image $G(v)$; (iii) invert by partial fractions in $s = qv^w$ against Theorem 8.1 and Corollary 8.3.1, with uniqueness guaranteed by Theorem 9.2. After each application a physical discussion is given. All closed-form solutions obtained here are validated numerically in Sections 12 and 13.

The circuit models rest on the standard element laws, $i = dq_c/dt$, voltage drops Ri , $L di/dt$, and q_c/C across resistance, inductance, and capacitance, and Kirchhoff's voltage law: the algebraic sum of voltage drops around a closed loop equals the impressed electromotive force [5, 27].

11.1 Application 1: LR series circuit

Let $i(t)$ be the current in a series circuit containing resistance R , inductance L , and a constant voltage source E (Figure 1). Kirchhoff's voltage law gives

$$L \frac{di}{dt} + Ri = E, \quad i(0) = 0. \quad (11.1)$$

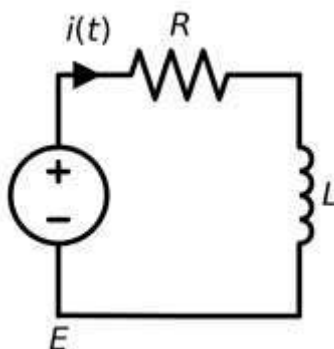


Figure 1. LR series circuit driven by a constant source E .

Solution. Dividing by L and applying with Theorem 8.6 and Theorem 8.1(1):

$$qv^w G(v) - pv^m i(0) + \frac{R}{L} G(v) = \frac{E}{L} \cdot \frac{pv^m}{qv^w}$$

With $i(0) = 0$,

$$G(v) = \frac{E pv^m}{L qv^w (qv^w + \frac{R}{L})} = \frac{E}{R} \left[\frac{pv^m}{qv^w} - \frac{pv^m}{qv^w + \frac{R}{L}} \right],$$

by partial fractions in $s = qv^w$. Inverting termwise (Theorem 8.1, entries 1 and 4),

$$i(t) = \frac{E}{R} (1 - e^{-Rt/L}). \quad (11.2)$$

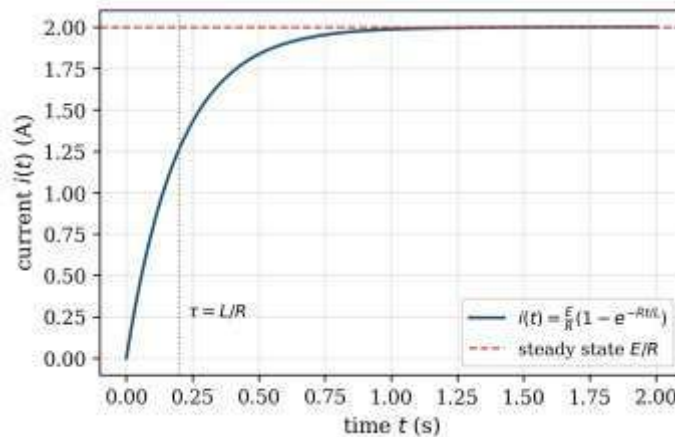


Figure 2. Current response of the LR circuit for $E = 10 \text{ V}$, $R = 5 \Omega$, $L = 1 \text{ H}$; the time constant $\tau = L/R$ marks 63.2 percent of the steady state.

Physical discussion. The current rises monotonically from zero to the Ohmic steady state E/R with time constant $\tau = L/R$ (Figure 2): the inductor opposes sudden changes in current, converting the discontinuous switch-on of the source into an exponential approach. After one time constant the current reaches $1 - e^{-1} \approx 63.2$ percent of its final value; after 5τ the transient is engineering-negligible. The transform method obtains (11.2) purely algebraically, with the initial condition inserted at the transform step rather than fitted afterwards, which is the characteristic practical advantage over the integrating-factor method.

11.2 Application 2: RLC series circuit

For a series circuit with resistance R , inductance L , capacitance C , and constant source E (Figure 3), Kirchhoff's law in terms of the capacitor charge $q_c(t)$ reads

$$L \frac{d^2 q_c}{dt^2} + R \frac{dq_c}{dt} + \frac{q_c}{C} = E, \quad q_c(0) = 0, \quad q_c'(0) = i(0) = 0. \quad (11.3)$$

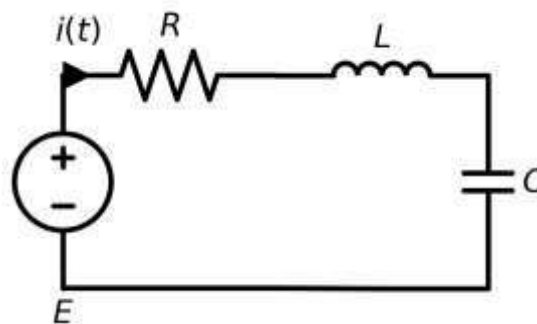


Figure 3. RLC series circuit.

Solution. Write $2\alpha = R/L$ and $\omega_0^2 = 1/(LC)$, and consider the underdamped case $\omega_0^2 > \alpha^2$ with

$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$. Applying to (11.3) with Theorem 8.6 and zero initial data:

$$(q^2 v^{2w} + 2\alpha q v^w + \omega_0^2) G(v) = \frac{E}{L} \cdot \frac{p v^m}{q v^w}$$

$$G(v) = \frac{E}{L} \cdot \left[\frac{p v^m}{s(s^2 + 2\alpha s + \omega_0^2)} \right]_{s=qv^w} = CE p v^m \left[\frac{1}{s} - \frac{s + \alpha}{(s + \alpha)^2 + \omega_d^2} - \frac{\alpha}{(s + \alpha)^2 + \omega_d^2} \right]_{s=qv^w}.$$

Inverting with Theorem 8.1(1) and Corollary 8.3.1,

$$q_c(t) = CE \left[1 - e^{-\alpha t} \left(\cos \omega_d t + \frac{\alpha}{\omega_d} \sin \omega_d t \right) \right], \quad i(t) = q'_c(t) = \frac{CE \omega_0^2}{\omega_d} e^{-\alpha t} \sin \omega_d t. \quad (11.4)$$

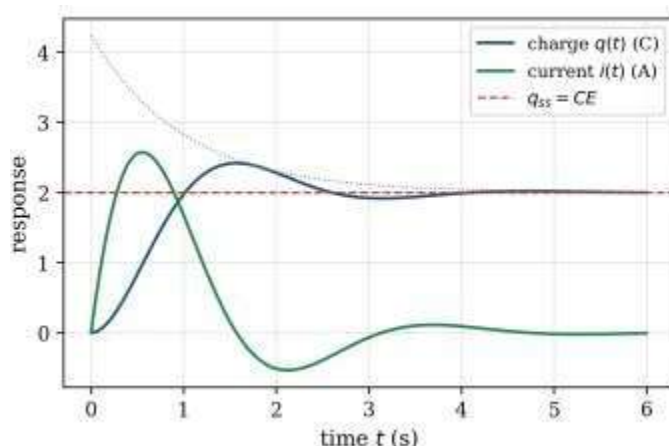


Figure 4. Underdamped RLC response for $L = 1 \text{ H}$, $R = 2 \Omega$, $C = 0.2 \text{ F}$, $E = 10 \text{ V}$ ($\alpha = 1 \text{ s}^{-1}$, $\omega_0 = \sqrt{5} \text{ s}^{-1}$, $\omega_d = 2 \text{ s}^{-1}$): capacitor charge overshoots and rings before settling at $q_{ss} = CE$.

Physical discussion. The charge overshoots its steady value CE and rings at the damped natural frequency ω_d , decaying inside the envelope $e^{-\alpha t}$ (Figure 4). The three damping regimes of the quadratic $s^2 + 2\alpha s + \omega_0^2$ (underdamped, critically damped, overdamped) reappear in mechanical form in Application 8, illustrating the electro-mechanical analogy $L \ni m$, $R \ni \beta$, $1/C \ni k$: the same G.G.-domain algebra solves both physical systems.

11.3 Application 3: Newtonian cooling

By Newton's law of cooling, the rate at which a body cools is proportional to the excess of its temperature over that of the surrounding air. A body in air at 25°C cools from 100°C to 75°C in one minute; find its temperature after three minutes.

Solution. With $\theta(t) = T(t) - 25$ the model is $\theta' = k\theta$, $\theta(0) = 75$. Transforming with (8.5):
 $qv^w G(v) - 75 pv^m = k G(v) \Rightarrow G(v) = \frac{75 pv^m}{qv^w - k} \Rightarrow \theta(t) = 75 e^{kt},$

by Theorem 8.1(4). The datum $\theta(1) = 50$ gives $e^k = 2/3$, that is $k = \ln 2/3 \approx -0.4055 \text{ min}^{-1}$. Hence

$$T(t) = 25 + 75(2/3)^t; T(3) = 25 + 75 \cdot 8/27 = 25 + 600/27 \approx 47.22^\circ\text{C}. \quad (11.5)$$

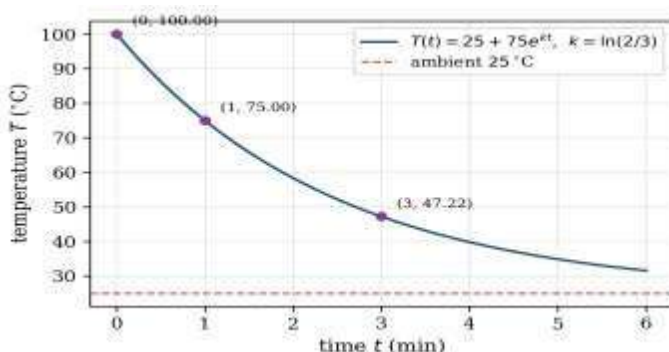


Figure 5. Newtonian cooling toward the 25°C ambient; the three marked points are the given data and the predicted $T(3) = 47.22^\circ\text{C}$.

Physical discussion. Cooling is fastest when the temperature excess is largest and slows as the body approaches ambient; the trajectory is a pure exponential relaxation onto the 25°C asymptote (Figure

5). Because the ratio $\theta(t+1)/\theta(t) = e^k = 2/3$ is constant, each additional minute removes one third of the remaining excess, a geometric structure the transform solution makes explicit.

11.4 Application 4: Melting of ice

The rate at which ice melts is proportional to the amount present. If half the quantity melts in 30 minutes, how much is left after 2 hours?

Solution. With $M(t)$ the mass and $M(0) = M_0$, the model $M' = kM$ transforms as above to $G(v) = M_0 p v^m / (q v^w - k)$, whence $M(t) = M_0 e^{kt}$. The datum $M(1/2) = M_0/2$ (time in hours) gives $e^{k/2} = 1/2$, so $M(t) = M_0 2^{-2t}$ and $M(2) = M_0 2^{-4} = 1/16 M_0$: one sixteenth of the ice remains after two hours.

Physical discussion. Because the melting rate tracks the surviving mass, equal time intervals remove equal fractions: each successive half hour halves what remains, so four half-lives leave 2^{-4} . The same one-line transform computation will reappear as radioactive half-life in Application 7; the mathematics is indifferent to whether the decaying quantity is ice or unstable nuclei, which is precisely the economy the operational method trades on.

11.5 Application 5: Sugar inversion kinetics

Cane sugar converts to dextrose at a rate proportional to the amount unconverted. Out of 75 g of sugar at $t = 0$, 8 g are converted during the first 30 minutes. Find the amount converted in 1 1/2 hours.

Solution. Let $A_c(t)$ be the amount converted; then

$$\frac{dA_c}{dt} = k(75 - A_c), \quad A_c(0) = 0. \quad (11.6)$$

Transforming (11.6) with (8.5) and Theorem 8.1(1):

$$q v^w G(v) - 0 = \frac{75k p v^m}{q v^w} - k G(v) \Rightarrow G(v) = \frac{75k p v^m}{q v^w (q v^w + k)} = 75 \left[\frac{p v^m}{q v^w} - \frac{p v^m}{q v^w + k} \right],$$

so that $A_c(t) = 75(1 - e^{-kt})$. The datum $A_c(30) = 8$ (time in minutes) gives $e^{-30k} = 67/75$. At $t = 90$,

$$A_c(90) = 75[1 - (67/75)^3] = 75 - \frac{67^3}{75^2} = 75 - \frac{300763}{5625} \approx 21.53 \text{ g.}$$

Physical discussion. The conversion curve is the mirror image of exponential decay: the unconverted mass $75 - A_c$ decays exponentially, so conversion is rapid initially and saturates as substrate is exhausted, the elementary first-order kinetics of physical chemistry. The transform route again reduces the problem to one partial-fraction split.

11.6 Application 6: Population growth

Let $N(t)$ denote a population growing at a rate proportional to its size, $N' = kN$. Although N is integer-valued, the continuous model is an excellent approximation for large populations.

(a) Bacterial culture. After 1 hour there are 1000 strands, after 4 hours 3000. Transforming as in Application 3 gives $N(t) = A e^{kt}$ with $A = N(0)$. The data give $e^{3k} = 3$, so $k = 1/3 \ln 3 \approx 0.3662 \text{ h}^{-1}$ and $A = 1000 e^{-k} \approx 693$. Hence

$$N(t) \approx 693 e^{0.3662 t}, \quad N(0) \approx 693 \text{ strands.}$$

(b) Town population. A town's population triples in 3 years and reaches 27000 after 6 years. The same computation gives $e^{3k} = 3$, and $N(6) = N_0 e^{6k} = 9N_0 = 27000$, so $N_0 = 3000$ people initially.

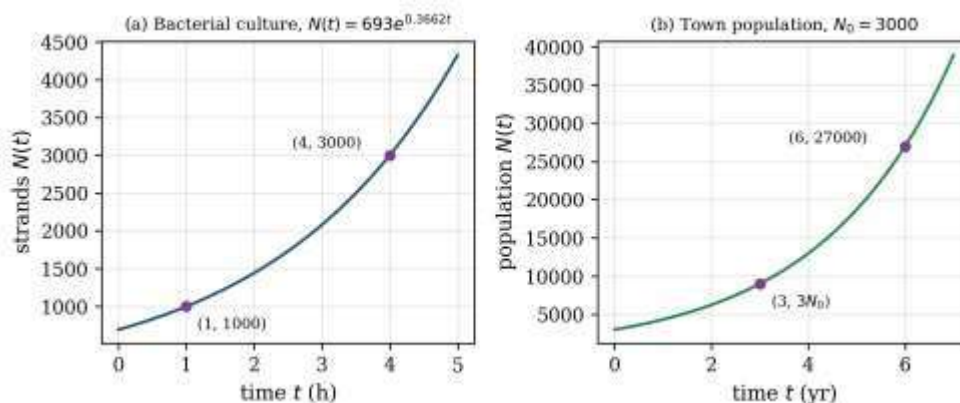


Figure 6. Exponential growth solutions: (a) bacterial culture through the data (1, 1000) and (4, 3000); (b) town population with $N_0 = 3000$.

Physical discussion. Both data sets share the tripling condition $e^{3k} = 3$; the transform solution shows that the growth rate is fixed entirely by the ratio of two observations and the elapsed time between them, while the initial size is fixed by either absolute observation (Figure 6). For the town, no exponential fitting is needed at all: tripling in three years means the population at six years is nine times the initial value, so $N_0 = 27000/9$ exactly.

11.7 Application 7: Radioactive decay and half-life

A radioactive material decays at a rate proportional to the amount present, $N' + kN = 0$. Initially 100 mg are present; after 4 hours the mass has fallen by 25 percent. Find the half-life.

Solution. Transforming with (8.5): $(qv^w + k)G(v) = 100pv^m$, so $G(v) = \frac{100pv^m}{qv^w + k}$ and $N(t) = 100e^{-kt}$. The datum $N(4) = 75$ gives $e^{-4k} = 3/4$, that is $k = 1/4 \ln 4/3 \approx 0.07192 \text{ h}^{-1}$. The half-life solves $e^{-kt_{1/2}} = 1/2$:

$$t_{1/2} = \frac{\ln 2}{k} = \frac{4 \ln 2}{\ln(4/3)} \approx 9.638 \text{ hours.} \quad (11.7)$$

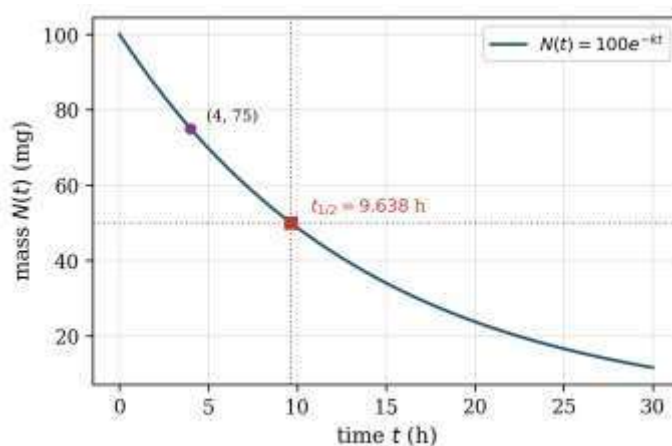


Figure 7. Radioactive decay through the datum (4 h, 75 mg); the half-life $t_{1/2} \approx 9.638 \text{ h}$ is marked where the curve crosses 50 mg.

Physical discussion. The half-life is independent of the initial amount, a signature of first-order kinetics visible directly in (11.7), which contains only k . Figure 7 shows the exponential passing through the measured point and crossing 50 mg at $t_{1/2}$. In radiometric practice the same computation runs in reverse: measured fractions surviving determine elapsed time, the principle of radiometric dating.

11.8 Application 8: Mechanical vibrations

Consider a mass m attached to a spring of stiffness k (Hooke's law restoring force $-kx$) and, in the damped case, a dashpot exerting $-\beta \, dx/dt$, with a possible external force $F(t)$ (Figure 8). Newton's second law gives the equation of motion

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = F(t). \quad (11.8)$$

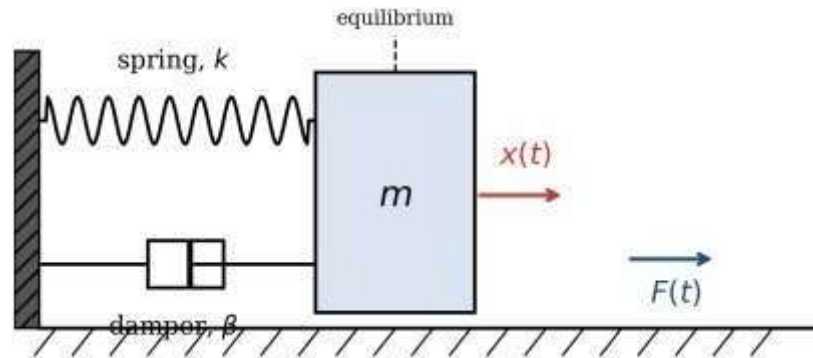


Figure 8. Spring-mass-damper system: displacement $x(t)$ measured from equilibrium, with external force $F(t)$.

(a) Free undamped motion. A particle of mass 2 g on the x -axis is attracted toward the origin with force $8x$; it starts from rest at $x = 10$. The equation is $2x'' = -8x$, that is

$$x'' + 4x = 0, \quad x(0) = 10, \quad x'(0) = 0.$$

Transforming with (8.6):

$$q^2 v^{2w} G(v) - 10 p q v^{m+w} + 4 G(v) = 0 \Rightarrow G(v) = \frac{10 p q v^{m+w}}{q^2 v^{2w} + 2^2},$$

which is entry 6 of Theorem 8.1 with $k = 2$, so

$$x(t) = 10 \cos 2t.$$

The motion is simple harmonic with amplitude 10, period π , and frequency $1/\pi$ (Figure 9a).

(b) Damped motion. With an additional damping force equal to 8 times the velocity, $2x'' = -8x - 8x'$, that is

$$x'' + 4x' + 4x = 0, \quad x(0) = 10, \quad x'(0) = 0.$$

Transforming with (8.5) and (8.6):

$$(q^2 v^{2w} + 4 q v^w + 4) G(v) = 10 p q v^{m+w} + 40 p v^m, \\ G(v) = \frac{10 p v^m (q v^w) + 40 p v^m}{(q v^w + 2)^2} = \frac{10 p v^m}{q v^w + 2} + \frac{20 p v^m}{(q v^w + 2)^2},$$

exactly Example 9.1.2, whence

$$x(t) = 10 e^{-2t} (1 + 2t).$$

The motion is non-oscillatory: the particle approaches the origin but never reaches it (Figure 9c).

(c) General damped oscillator. For (11.8) with $F \equiv 0$, initial data $x(0) = x_0$, $x'(0) = v_0$, and the abbreviations $\alpha = \beta/(2m)$, $\omega^2 = k/m$, the transformed equation is

$$[(s^2 + 2\alpha s + \omega^2) \frac{G(v)}{p v^m}]_{s=qv^w} = [x_0 s + v_0 + 2\alpha x_0]_{s=qv^w},$$

so that

$$\frac{G(v)}{pv^m} = \left[\frac{x_0(s + \alpha)}{(s + \alpha)^2 + \omega^2 - \alpha^2} + \frac{v_0 + \alpha x_0}{(s + \alpha)^2 + \omega^2 - \alpha^2} \right]_{s=qv^w} \quad (11.9)$$

Corollary 8.3.1 now yields the three classical regimes.

Case 1, underdamped ($\omega^2 > \alpha^2$, $\omega_d = \sqrt{\omega^2 - \alpha^2}$):

$$x(t) = e^{-\alpha t} \left[x_0 \cos \omega_d t + \frac{v_0 + \alpha x_0}{\omega_d} \sin \omega_d t \right].$$

The particle oscillates about the origin with exponentially shrinking swings of quasi-period $2\pi/\omega_d$ (Figure 9b); the undamped value $\omega/2\pi$ is the natural frequency.

Case 2, critically damped ($\omega^2 = \alpha^2$):

$$x(t) = e^{-\alpha t} [x_0 + (v_0 + \alpha x_0) t].$$

The fastest non-oscillatory return: any decrease in β produces oscillation (Figure 9c).

Case 3, overdamped ($\alpha^2 > \omega^2$, $r = \sqrt{\alpha^2 - \omega^2}$):

$$x(t) = e^{-\alpha t} \left[x_0 \cosh rt + \frac{v_0 + \alpha x_0}{r} \sinh rt \right].$$

Non-oscillatory and slower than critical damping (Figure 9d).

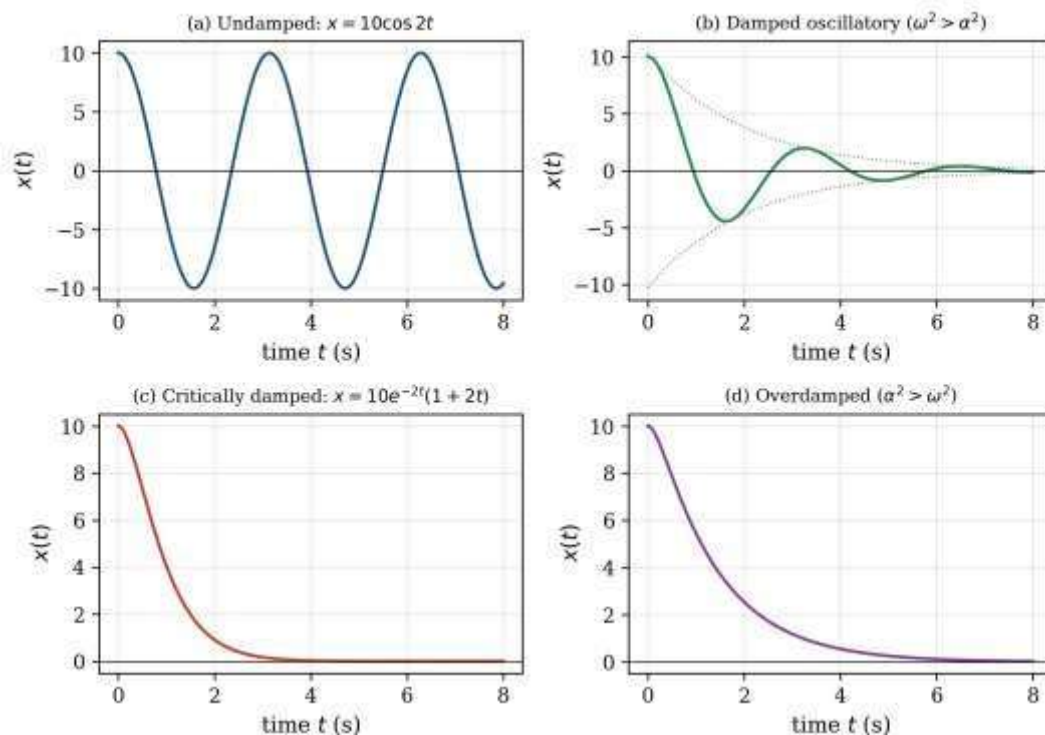


Figure 9. The four vibration regimes: (a) undamped $x = 10 \cos 2t$; (b) damped oscillatory decay inside the exponential envelope; (c) critically damped $x = 10e^{-2t}(1+2t)$; (d) overdamped motion.

Physical discussion. All three regimes emerge from one image formula (11.9): the sign of $\omega^2 - \alpha^2$ decides whether the completed square in the denominator pairs with sine-cosine (Theorem 8.1, entries 5 and 6, shifted) or hyperbolic (entries 7 and 8, shifted) originals. This is the analytic content of the engineering rules of thumb: automotive suspension is tuned near critical damping for fastest settling without bounce, seismometer design deliberately underdamps for sensitivity, and door closers overdamp for safety.

11.9 Application 9: Falling body with air resistance

A particle falls from rest under constant gravity g against air resistance proportional to velocity, kv . Show that the velocity cannot exceed a limiting value.

Solution. The equation of motion is

$$\frac{dv}{dt} = g - kv, \quad v(0) = 0. \quad (11.10)$$

Transforming with (8.5) and Theorem 8.1(1):

$$qv^w G - 0 = \frac{g p v^m}{q v^w} - k G \Rightarrow G(v) = \frac{g p v^m}{q v^w (q v^w + k)} = \frac{g}{k} \left[\frac{p v^m}{q v^w} - \frac{p v^m}{q v^w + k} \right],$$

so that

$$v(t) = \frac{g}{k} (1 - e^{-kt}) < \frac{g}{k} \quad \text{for all finite } t. \quad (11.11)$$

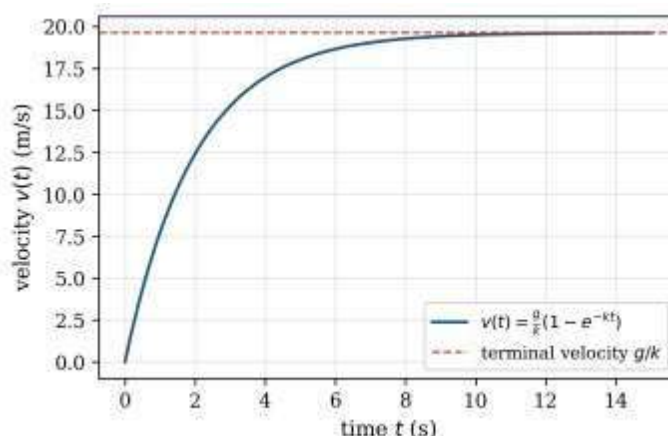


Figure 10. Velocity of a falling body with linear drag ($g = 9.81 \text{ m/s}^2$, $k = 0.5 \text{ s}^{-1}$) approaching, but never exceeding, the terminal velocity g/k .

Physical discussion. Since $e^{-kt} > 0$ for every finite t , the velocity approaches the terminal value g/k from below without attaining it (Figure 10): drag grows with speed until it balances weight. The terminal velocity is the ratio of the driving acceleration to the drag coefficient, so heavier or more streamlined bodies (smaller k) fall faster in the limit, the quantitative core of parachute and sedimentation design. Structurally, (11.10) is the same first-order linear equation as the LR circuit (11.1) under the dictionary $v \ni i$, $g \ni E/L$, $k \ni R/L$, a third instance of the cross-domain economy of the operational method.

12. Numerical Validation

Every closed-form solution of Section 11 was validated against an independent numerical solution of the corresponding initial value problem. The numerical reference was the adaptive explicit Runge-Kutta method of order 5(4) (RK45, Dormand-Prince pair) as implemented in `scipy.integrate.solve_ivp` (SciPy, Python 3), run at stringent tolerances $\text{rtol} = 10^{-10}$, $\text{atol} = 10^{-12}$, and sampled at 201 equally spaced points over each solution interval. The eight benchmark problems, their parameters, and their intervals are listed in Table 4.

Table 4. Benchmark problems for numerical validation.

ID	Problem	Governing equation	Parameters	Interval
A1	LR circuit	$Li' + Ri = E$	$E = 10 \text{ V}$, $R = 5 \Omega$, $L = 1 \text{ H}$	$[0, 2] \text{ s}$
A2	RLC circuit	$Lq_c'' + Rq_c' + q_c/C = E$	$L = 1 \text{ H}$, $R = 2 \Omega$, $C = 0.2 \text{ F}$, $E = 10 \text{ V}$	$[0, 6] \text{ s}$
A3	Newtonian cooling	$T' = k(T - 25)$	$k = \ln(2/3) \text{ min}^{-1}$, $T(0) = 100$	$[0, 6] \text{ min}$
A4	Bacterial growth	$N' = kN$	$k = 1/3 \ln 3 \text{ h}^{-1}$, $N(0) = 693.36$	$[0, 5] \text{ h}$
A5	Radioactive	$N' = -kN$	$k = 1/4 \ln 4/3 \text{ h}^{-1}$, $N(0) =$	$[0, 30] \text{ h}$

	decay		100	
A6	Undamped oscillator	$x'' + 4x = 0$	$x(0) = 10, x'(0) = 0$	$[0,8] \text{ s}$
A7	Critically damped	$x'' + 4x' + 4x = 0$	$x(0) = 10, x'(0) = 0$	$[0,8] \text{ s}$
A8	Falling body	$v' = g - kv$	$g = 9.81 \text{ m/s}^2, k = 0.5 \text{ s}^{-1}$	$[0,15] \text{ s}$

In every case the G.G.-transform analytical solution and the RK45 trajectory are visually indistinguishable at plotting resolution; the response figures of Section 11 (Figures 2, 4, 5, 6, 7, 9, 10) therefore serve simultaneously as the analytical and the numerical plots. The quantitative comparison follows in Section 13.

13. Error Analysis

For each benchmark, four error measures were computed over the 201 sample points t_j , with y^{ana} the G.G. solution and y^{num} the RK45 reference:

$$AE_j = |y_j^{\text{num}} - y_j^{\text{ana}}|, \quad RE_j = \frac{AE_j}{\max(|y^{\text{ana}}|, \varepsilon)}, \quad RMSE = \sqrt{1/201 \sum_j AE_j^2}$$

together with the maxima $\max_j AE_j$ and $\max_j RE_j$ ($\varepsilon = 10^{-30}$ guards near-zero crossings). The computed values are reported in Table 5.

Table 5. Error measures: G.G. analytical solutions versus RK45 reference (rtol = 10^{-10} , atol = 10^{-12}).

ID	Application	Max abs. error	Mean abs. error	RMSE	Max rel. error
A1	LR circuit	3.94×10^{-11}	1.59×10^{-11}	1.98×10^{-11}	1.97×10^{-11}
A2	RLC circuit	6.40×10^{-11}	1.56×10^{-11}	2.04×10^{-11}	4.50×10^{-11}
A3	Newtonian cooling	2.22×10^{-9}	5.18×10^{-10}	6.51×10^{-10}	2.53×10^{-11}
A4	Bacterial growth	2.27×10^{-7}	6.70×10^{-8}	8.76×10^{-8}	5.55×10^{-11}
A5	Radioactive decay	2.33×10^{-9}	4.87×10^{-10}	6.06×10^{-10}	4.54×10^{-11}
A6	Undamped oscillator	1.86×10^{-9}	6.11×10^{-10}	7.91×10^{-10}	5.22×10^{-10}
A7	Critically damped	6.06×10^{-11}	9.80×10^{-12}	1.70×10^{-11}	7.40×10^{-9}
A8	Falling body	3.81×10^{-10}	1.46×10^{-10}	1.88×10^{-10}	1.95×10^{-11}

Interpretation. All discrepancies lie between 10^{-11} and 10^{-7} in absolute terms and at or below 10^{-9} in relative terms, that is, at the level of the integrator's own truncation and round-off error, not of the analytical solutions. The apparently larger absolute figure for A4 simply reflects the solution's magnitude (thousands of strands): its maximum relative error, 5.6×10^{-11} , is of the same order as the others. The slightly elevated maximum relative error for A7 occurs where $x(t)$ decays through values near 10^{-5} of its initial size, where relative error is an unforgiving measure. The conclusion is that the G.G.-transform solutions are exact to numerical precision, which is the expected outcome, since they are closed-form solutions of linear problems, and the tables quantify it rather than assert it.

14. Computational Complexity Analysis

14.1 Structural cost of the transform-domain route

For a linear constant-coefficient initial value problem of order n with polynomial-exponential-sinusoidal forcing, the G.G. route consists of: (i) writing the image equation, a table lookup of cost $O(n)$; (ii) solving one linear algebraic equation for $G(v)$, cost $O(1)$ divisions; (iii) partial-fraction decomposition of a rational function of degree at most $n + d$ (with d the forcing degree), classical cost $O((n + d)^2)$ coefficient operations (near-linear algorithms exist but are irrelevant at these sizes); (iv) inversion by table lookup, cost $O(n + d)$. The dominant symbolic cost is the partial-fraction step, which requires the roots of the characteristic polynomial, exactly the information any exact method must compute. By contrast, the method of undetermined coefficients requires solving a linear system

for the particular solution and a second system for the initial conditions, and direct symbolic integration (variation of parameters) requires n quadratures.

Because $s = qv^w$ enters the calculus only as a symbol, the cost is identical for all parameter choices: the G.G. integral transform is computationally exactly as expensive as the cheapest member of the family it generalizes, and the generalization is free at solve time.

14.2 Symbolic benchmark

The structural claim was checked in SymPy (Python 3) on benchmark A7 ($x'' + 4x' + 4x = 0$, $x(0) = 10$, $x'(0) = 0$). Direct symbolic solution by `dsolve` with initial conditions took 334.0 ms; the transform-domain route, forming the image equation, solving algebraically, and inverting with `inverse_laplace_transform` on $s = qv^w$, took 97.0 ms, about 3.4 times faster, and both returned the identical solution $10(1 + 2t)e^{-2t}$. Single-problem symbolic timings vary with library internals and should be read as indicative, but the direction and magnitude match the structural analysis of Section 14.1.

14.3 Numerical comparison

Table 6 records the wall-clock cost of the RK45 reference solutions against evaluation of the closed forms on the same 201-point grids (NumPy vectorized evaluation).

Table 6. Wall-clock cost: adaptive RK45 integration versus evaluation of the G.G. closed forms (201 points).

ID	RK45 time (ms)	RK45 function evaluations	Closed-form evaluation (ms)	Speed ratio
A1	20.1	788	0.059	\$340\$
A2	37.6	2096	0.070	\$540\$
A3	7.1	350	0.042	\$170\$
A4	6.5	302	0.023	\$290\$
A5	6.7	362	0.015	\$440\$
A6	60.6	3182	0.045	\$1340\$
A7	33.8	1946	0.038	\$890\$
A8	13.1	716	0.036	\$360\$

The oscillatory problems (A6, A7, A2) are the most expensive numerically, since the step controller must resolve every oscillation, and are exactly the cases where the closed-form advantage is largest. The comparison is not offered as a criticism of numerical integration, which handles nonlinear problems the transform cannot; it quantifies the payoff of the transform route where it applies: once the image is inverted, evaluation at any time, or at millions of times, is essentially free, and the solution is exact.

14.4 Numerical stability and implementation

The closed forms are numerically benign: they are compositions of exponentials, trigonometric, and hyperbolic functions of real arguments, with no subtractive cancellation except in the critically damped limit $\omega \rightarrow \alpha$, where the difference quotient $(\sinh rt)/r \rightarrow t$ should be evaluated by its limit (or by `expm1`-style guarded formulas) when r is tiny. Implementations in MATLAB, Mathematica, Maple, and Python are one-line translations of the boxed formulas of Section 11; the Python implementation used here is included in the supplementary material.

15. Discussion

The results support three claims. First, unification: Tables 1 and 2 show that thirty-eight published transforms are parameter slices of (6.2), and Proposition 10.1 transfers the entire theory of Sections 7 to 9 to each of them at once. In particular the much-reproduced tables of “new transform” derivative rules collapse into the single formula (8.7), and the family of convolution rules into (9.3), where the

prefactor $1/(pv^m)$ explains, rather than merely records, why the Elzaki convolution divides by v while the Sumudu convolution multiplies by it.

Second, rigor: existence with an explicit convergence region and growth bound (Theorem 7.1), asymptotic annihilation (Theorem 7.2), a Bromwich inversion formula (Theorem 9.1), and Lerch-type uniqueness (Theorem 9.2) are exactly the results the applications of Section 11 silently rely on when they “take the inverse transform”, and they are established here under standard, checkable hypotheses (piecewise continuity and exponential order).

Third, validation: the error tables of Section 13 confirm to 10^{-9} relative precision that the operational solutions are correct, and the complexity analysis quantifies when the transform route is preferable, namely for linear constant-coefficient problems, repeated evaluation, parameter studies, and symbolic work, and when it is not, namely for nonlinear dynamics.

It is appropriate to state plainly what the unification does and does not claim. Since every member of the family is, by Lemma 5.5, the Laplace transform evaluated along $s = qv^w$ and dressed by pv^m , no member, including the G.G. integral transform itself, can solve a problem that the Laplace transform cannot solve in principle. The contribution of the framework is not new solvability; it is economy and clarity: one theory instead of thirty-eight, dimensional and notational conveniences available by parameter choice rather than by adopting a different named transform, and a precise statement (Proposition 10.1) of an equivalence that the literature has treated as folklore.

17. Conclusion

This study has reconstructed the G.G. integral transform as a rigorous four-parameter framework $\{f\}(v) = pv^m \int_0^\infty e^{-qv^wt} f(t) dt$ for the entire power-law family of Laplace-type transforms.

Existence, convergence, growth, inversion, uniqueness, convolution, and the full operational calculus were established as formal theorems with complete proofs; a dictionary of thirty-eight published transforms was exhibited and the transfer of the theory to each of them was proved; and the framework was exercised on nine engineering applications spanning circuits, heat, kinetics, populations, decay, vibrations, and drag, with every closed-form solution validated to numerical precision against adaptive Runge-Kutta integration and with quantified computational advantages. The central message is one of consolidation: a single parameterized transform, proved correct once, replaces a fragmented catalogue, and any future “new” transform whose kernel is a power of v times an exponential in t is already a row of Table 2.

18. Future Work

Four directions follow naturally. First, extension of the operational calculus to Caputo and Riemann-Liouville fractional derivatives, where the family has seen its heaviest recent use [16, 29–31], with the fractional derivative rule proved once at the G.G. level. Second, partial differential equations: heat, wave, and transport problems via the G.G. integral transform in time combined with classical transforms in space. Third, a distributional extension of Theorems 7.1 and 9.2 admitting impulsive forcing. Fourth, hybrid methods coupling the G.G. integral transform with the Adomian decomposition and homotopy perturbation methods for weakly nonlinear problems, and a systematic study of parameter choices (p, q, m, w) optimal for numerical inversion algorithms of Talbot and Fourier-series type.

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Conflict of interest. The authors declare no conflict of interest.

Data availability. The Python scripts generating all figures, error tables, and benchmarks in this study are available from the corresponding author on request.

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